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THE EFFECT OF THE SHAPE OF THE INCOME DISTRIBUTION ON TWO INEQUALITY MEASURES *

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This letter describes the effect of skewness and kurtosis of the log-income distribution on two measures of income inequality. Positive skewness and leptokurtosis affect inequality positively.

1. Description of the result

Theil (1967) proposed two measures of income inequality, both based on statistical information theory. One measure, J_y, is the expected information of the message which transforms the population shares into the income shares; the other, J_p, has the same interpretation except that the roles of the population and income shares are reversed. Since Bourguignon (1979) has shown that J_p and J_y are the only inequality measures which satisfy an additive decomposability axiom, a further analysis of these two measures is appropriate. It is known [1] that if the income distribution is lognormal, both J_p and J_y are equal to $\frac{1}{2}\sigma^2$, where σ^2 is the variance of the logarithm of income. How is this result affected when the income distribution deviates from lognormality?

The simplest way to answer this question is by expressing J_y and J_p in terms of income z,

$$J_y = \mathcal{E}\left(\frac{z}{\mathcal{E}z} \log_e \frac{z}{\mathcal{E}z}\right), \qquad J_p = \mathcal{E}\left(\log_e \frac{\mathcal{E}z}{z}\right). \tag{1}$$

We write $y = \log_e z$, $\mathcal{E}y = \mu$, [2] $\operatorname{var} y = \sigma^2$, and introduce the skewness coefficient

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[1] See Theil (1967, ch. 4), also for the proof of eq. (1) below.

[2] Hence e^{μ} is geometric mean income. The existence of a finite geometric mean income requires that the (non-logarithmic) income distribution have zero density at zero income.

$\gamma_1 = \mathcal{E}(y - \mu)^3/\sigma^3$ of the log-income distribution and its kurtosis coefficient $\gamma_2 = \mathcal{E}(y - \mu)^4/\sigma^4 - 3$. In section 2 we prove the following result:

$$J_y = \tfrac{1}{2}\sigma^2 [1 + \tfrac{2}{3}\gamma_1 \sigma + \tfrac{1}{4}\gamma_2 \sigma^2 + o(\sigma^2)] , \tag{2}$$

$$J_p = \tfrac{1}{2}\sigma^2 [1 + \tfrac{1}{3}\gamma_1 \sigma + \tfrac{1}{12}\gamma_2 \sigma^2 + o(\sigma^2)] . \tag{3}$$

Thus, when the log-income distribution is positively skewed (with a long tail on the right), both J_p and J_y exceed the lognormal value $\frac{1}{2}\sigma^2$ and the excess is twice as large for J_y as it is for J_p. Similarly, when the log-income distribution is leptokurtic (i.e., when $\gamma_2 > 0$), both J_p and J_y exceed the lognormal value $\frac{1}{2}\sigma^2$. The latter result is intuitively easy to understand, because leptokurtic distributions have relatively fat tails which should positively contribute to inequality.

2. Derivations

We write $\mathcal{E}(\log_e z) = y = \mu = 0$; this is simply a matter of selecting an appropriate income unit. We can then write J_p of (1) as

$$J_p = \log_e \mathcal{E}(e^y) . \tag{4}$$

A Taylor expansion of e^y yields

$$\mathcal{E}(e^y) = 1 + \tfrac{1}{2}\sigma^2 + \tfrac{1}{6}\gamma_1 \sigma^3 + \tfrac{1}{24}(\gamma_2 + 3)\, \sigma^4 + o(\sigma^4) \tag{5}$$

after which the proof of (3) is completed by a Taylor expansion of $\log_e \mathcal{E}(e^y)$ in accordance with (4).

If $\mu = 0$, we can write J_y of (1) as

$$J_y = -J_p + \frac{\mathcal{E}(ye^y)}{\mathcal{E}(e^y)} . \tag{6}$$

A Taylor expansion of ye^y yields

$$\mathcal{E}(ye^y) = \sigma^2 + \tfrac{1}{2}\gamma_1 \sigma^3 + \tfrac{1}{6}(\gamma_2 + 3)\, \sigma^4 + o(\sigma^4) .$$

By multiplying this by $1/\mathcal{E}(e^y) = 1 - \frac{1}{2}\sigma^2 + o(\sigma^2)$ [see (5)] we obtain

$$\frac{\mathcal{E}(ye^y)}{\mathcal{E}(e^y)} = \sigma^2 + \tfrac{1}{2}\gamma_1 \sigma^3 + \tfrac{1}{6}\gamma_2 \sigma^4 + o(\sigma^4) .$$

The proof of (2) is then completed by means of (6) and (3).

A sufficient condition for the convergence of the expansions (2) and (3) is that the log-income distribution has a variance σ^2 less than 1 (which may be violated when the distribution is very 'unequal') and finite moments of every order. See Sargan (1974, 1976) and Srinivasan (1970) for discussions of moment expansions.

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