Multidimensioned Approaches to Welfare Analysis

By

Esfandiar Maasoumi
Department of Economics, SMU, Dallas, TX 75275-0496
MAASOUMI@mail.smu.edu
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Abstract: A general survey of multidimensional approaches to the measurement and comparisons of welfare situations is provided. The survey covers some aspects of welfare representation, data availability and experience, axiomatic characterization of inequality measures, existing multidimensional indices, the stochastic dominance alternative for comparing welfare situations, and mobility. Applications to national and international data are described. The paper emphasizes implementability issues and statistical methods in the context of a comparative analysis of the philosophical and theoretical desirability of alternative approaches.

1. Introduction

Multidimensioned consideration of welfare and inequality is arguably one of the most pressing issues facing the credible *practice* of welfare analysis. The very meaning of "income" inequality is ambiguous when households and individuals are known to have different characteristics and needs. Life-cycle differences in incomes is another impediment to a meaningful conceptualization of "income inequality". And non-monetizable or non-tradable benefits that affect well being are non-income dimensions worthy of inclusion and/or independent study. These issues raise serious questions of legitimacy of an applied income-based univariate approach to welfare comparisons. Policy makers and others interested in optimal decision making often compare welfare situations and uncertain outcomes that involve many inherent characteristics. Suitable criteria as well as proper characterization and measurement of each "welfare situation" are required. These requirements are inter-related and difficult to meet.

Welfare criteria have been typically single dimensioned in the sense of being individualistic and utilitarian with only *homogenous* individual (or household) utility functions as their arguments. These utilities are replaced by incomes representing indirect utilities of optimizing units. The important question of heterogeneity among individuals is often inadequately dealt with. Realistic welfare characterization requires agreement on which attributes are necessary for inclusion, as well as their weights, if any, and their interactions. This multiattribute or heterogeneous context makes interpersonal comparisons of welfare, and its attendant problems, inevitable. It is for this reason that until quite recently income, perhaps adjusted by "equivalence scales", and its distribution had dominated in a "univariate approach" to applied and theoretical welfare analysis. A similar situation predominates in the finance literature.

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It is true that, in many societies, the income "proxy" is useful beyond its representation of consumption bundles, and it may proxy for other opportunities and rights. But income is inadequate to various degrees in different societies and at different times since it may not fully reflect all the benefits that people receive, nor their "needs" and their "abilities". Many of the benefits of education, public services and amenities, and quality of life attributes cannot even be priced for lack of market transaction. But data are increasingly available on non-income attributes that are meaningful both directly, in terms of their contribution to well-being, and by their characterization of characteristics that represent heterogeneity in such welfare units as individuals, households, and countries.

These limitations have been pointed out by several authors including Arrow (1971), Sen (1970, 1973), Kolm (1977), Atkinson and Bourguignon (1982), and Maasoumi (1979, 1986). Kolm (1977) appealed to "fundamentalism" and "impartiality" which imply that the greater the number of welfare attributes considered the more reasonable are the common assumptions of **anonymity** and homogeneity. Sen (1970, 73, 1977) pointed to the differences in needs and argued against the purely individualistic and utilitarian characterization of welfare. Some otherwise socially desirable "entitlements" and "abilities" are examples of attributes that are generally not evaluated within individual utility functions, or by an individualistic/utilitarian social welfare function; also see Arrow (1971).

Atkinson and Bourguignon (1982, 1987, 1989), Maasoumi (1986a, 1989a), Maasoumi and Jeong (1985), for example, deal with the theoretical and practical implementation issues, such as identification of desirable multidimensional indices, aggregation issues, conditions for ordering multivariate welfare functions. Maasoumi and Zandvakili (1986, 1989, 1990) discover a different source of variation across time and individuals. Variation in earnings is partly reflective of individuals' states in their life-cycles. An adjustment for this may be made by looking at the distribution of "permanent" income. A proper treatment of this will have to take into account the intertemporal aspects of decision making and treat incomes at different points as distinct attributes. This was done by Maasoumi and Zandvakili (1986) and applied in several empirical applications to US data.

"Double counting" and clustering solutions, and other data related questions have been addressed in an expanding array of studies, such as Hirschberg, Maasoumi, and Slottje (1991), Maasoumi and Nickelsburg (1988), and Atkinson, Bourguignon, and Morrison (1992). In the theoretical domain, Tsui (1992a, 1992b), U. Ebert (1995a, 1995b), Dardanoni (1992), and Shorrocks (1995) have further advanced the multidimensional welfare theory.

Even if agreement is reached on substantive questions regarding the practical meaning and measurement of suitable attributes to be included in the characterizations, the question of consensus functionals and criteria for evaluating welfare situations endures. Substantial and substantive disagreements about this question, albeit sometimes ill-expressed or implicit, are at the heart of the problem of *consensus* rankings and policy analysis. But the inevitability of the multidimensional welfare comparisons derives from the nature of the activity. It is this inevitability which renders this area of welfare analysis the centerpiece of future theoretical and empirical developments even if we have to live with what I will refer to as "majority" rankings.

The most common practical approach to *measurement* and comparing welfare situations is based on indices. For example, before and after-tax income distributions, or incomes of different regions, or the same region but across time, are explicitly and implicitly ranked by scalar measures such as inequality or poverty indices. There is something rather inevitable about this activity which arises out of the practical need for "measurement" as compared with "ordering". The theoretical methods for identifying "ideal" indices of inequality or poverty in the single attribute case are now well developed. These are summarized in this paper; also see Maasoumi (1996a). This "index-based" approach is limited in several respects, the most serious being a lack of consensus on the acceptable index even when there is broad, or majority, agreement about certain normative principles. This difficulty is a consequence of the lack of consensus with regards to the appropriate cardinalization of the class of admissible welfare functions. To provide for some degree of continuity and sufficiency, I will give a brief account of some salient theoretical findings in this approach.

As might be expected, the issue of consensus characterization and indexing looms even larger in the multidimensional analysis. Here, a welfare matrix of several attributes characterizing, or enjoyed by, welfare units (households) constitutes the argument of a "welfare function". The axioms under which a suitable index (cardinalization) may be identified are even less likely to command general agreement, and are less likely to produce reasonably small families of suitable scalar measures.

There are two basic methods in existence that generate "ideal" multivariate indices. The "two-step" method proposed in Maasoumi (1986a) obtains ideal aggregation functions over the desired attributes in its first step. The second step invokes the same fundamental welfare axioms that have been acceptable in the univariate/utilitarian setting which identify well known index families (such as the Generalized Entropy). This approach has a long history since it is implicitly adopted whenever we analyze equivalent incomes or costs of living for heterogenous households in place of their ordinary incomes.

The second method seeks to identify a set of properties (axioms) directly suited to the multivariate setting, and obtains the index (es) which may satisfy these axioms. Predictably, the identification of a commonly acceptable set of axioms has proven to be more ellusive than in the univariate case. I will provide an account of the related theoretical and empirical findings in this multivariate setting.

When "ordering" different states is our only goal, stochastic dominance and other orderings offer a philosophically attractive alternative to the seemingly daunting task of developing "ideal" indices. This is already so in the univariate approach since even a set of commonly adopted "fundamental welfare axioms" is only able to produce a large family of candidate welfare functions, capable of supporting an even larger set of inequality and other indices. Thus the possibility of avoiding cardinalization is even more alluring in the multidimensional case. As with the index approach, however, there are as yet unresolved philosophical and practical problems. These issues are discussed in this paper with an eye for *statistical* decision making and empirical implementation.

In what follows I will briefly review some elements of the univariate analysis which help to set the multidimensional issues in context (section 2). Then I will give an account of the multidimensional characterization and index development in section 3. This section will contrast the "two step" approach of this author with the "one step", or direct approach to identifying multivariate indices. In section 4 an account is given of the stochastic dominance approach and the statistical advances that have been made in very recent times. Section 5 includes some empirical examples of both approaches. An extensive, but by no means comprehensive, bibliography ends the paper.

2. Univariate Indices

2.1. **Inequality Measures and SWFs.** It is of general interest to compare distributions across countries, or over time for the same region, perhaps in order to assess any impact of economic policy or events. This requires that we estimate such things as Lorenz and Generalized Lorenz (GL) curves in order to evaluate any dominance relations statistically.

It is also an independently interesting and common practice to compare distributions on the basis of specific inequality indices. Indeed, it is very common to explicitly or implicitly rank income distributions on the basis of these estimated inequality measures without proceeding to a comparison of Lorenz-type curves. This is necessary when complete ranking is desired.

2.1.1. ***Measures of Inequality.*** Kolm (1969) and Atkinson (1970) have provided clear and influential formalizations of the relationship between SWFs and inequality measures[2] . Since then there has been much progress in both expanding on this important relationship and in utilizing it for more informed analyses of inequality measures. Kolm and Atkinson considered a utilitarian and individualistic welfare function which was increasing in incomes and equality preferring.

Let X be the income variable, μ_x its mean, and X_e the "equal equivalent income"; *i.e.*, the level of income which if received by everyone would leave social welfare at the same level as for a given income vector. Thus $X_e < \mu_x$ so long as there is any inequality, and a measure of divergence between these two would indicate the degree of welfare loss due to inequality. Atkinson (1970) and Kolm (1969) argued that, see also Blackorby and Donaldson (1978),

$$I(X) = 1 - X_e/\mu_x \tag{2.1}$$

may be a good measure of "relative" inequality. Indeed, one could just as well take:

$$I(X) = 1 - SWF(X)/\mu_x \tag{2.2}$$

where $SWF(X)$ is the "average" or mean Social Welfare Function (SWF). This should make clear that the SWF approach does not by itself identify a unique inequality index even when a particular SWF is agreed upon.

An important example of the measures generically defined above is the Atkinson family of inequality measures :

$$A_v = 1 - [\int_0^\infty x^{1-v} dF]^{\frac{1}{1-v}}/\mu_x, v > 0, v \neq 1 \tag{2.3}$$

[2] See Cowell and Kuga (1981) and also C. Dagum(1993) for even earlier contributions. The idea for a modern SWF discipline is attributed to Dalton of the London School of Economics in the 1920s. Interestingly, however, its revival is anticipated in **econometric** work which is contemporaneous with Kolm's theoretical writings in the middle 60s; see Aigner and Heins (1967)!

$$= 1 - exp[\int \log(x/\mu_x)dF], v = 1 \tag{2.4}$$

where F is the $c.d.f$ of income. Similarly, the Generalized Entropy (GE) family of indices is given by:

$$I_\gamma(X) = \frac{1}{\gamma(1+\gamma)} \int_0^\infty (x/\mu_x)[(x/\mu_x)^\gamma - 1]dF, \gamma \, \text{real} \tag{2.5}$$

I_1 is ordinally equivalent to the coefficient of variation and the Herfindahl index, and the family includes the variance of logarithms and Theil's first and second measures, I_0 and I_{-1}, respectively. Also, up to a monotonic transformation, there is a unique member of GE corresponding to each member of the Atkinson family. $v = -\gamma$ is the degree of aversion to relative inequality; the higher its absolute value the greater is the sensitivity of the measure to inequality (transfers) in the tail areas of the distribution.

The axiomatic derivation technique that identifies GE is constructive and is to be appreciated as an important breakthrough in organizing learning and knowledge in this area. It will help to set the multivariate issues in context. This axiomatic approach owes much to functional analysis first developed in "information theory", see Maasoumi (1993). In this approach one puts down an **explicit** set of requirements (axioms) which the ideal indices must satisfy, and which may or may not be universally acceptable. Using these axioms as explicit constraints on the function space one then obtains the appropriate inequality index. To exemplify, let us follow Bourguignon (1979) or Shorrocks (1980, 1984) in their discussion of the "fundamental welfare axioms" of symmetry, continuity, Principle of Transfers, and additive decomposability which identify GE as the desirable scale invariant family of **relative** inequality measures.

Axiom 1. The inequality index (function) is symmetric in incomes.

This is equivalent to anonymity which requires that only income matters not the identity of its recipient.

Axiom 2. Principle of transfers holds.

This requires that inequality decrease if we redistribute from a single richer individual to a poorer one, leaving their respective ranking and all the other individuals' incomes unchanged.

Axiom 3. Continuity.

This is relatively innocuous, helping in the mathematical derivations and in comparing different populations. None of the well known inequality indices violates this requirement. Note that it does allow for the practically non-sensical zero inequality.

Axiom 4. Invariance to scalar multiplication.

This is a serious limitation as it restricts attention to "relative" inequality. This is so since this requirement implies mean invariance; doubling everyone's income would leave inequality unchanged. Questions of "efficiency" can only be taken up by absolute inequality measures. None of the popular measures violates this requirement!

The class of functionals satisfying Axioms 1-4 is still too large. Also, any further axioms are less likely to command consensus. In fact, any further requirements must be justified by plausible considerations of such things as policy, empirical necessity, and practical interest. The most commonly invoked of such requirements is:

Axiom 5. Additive decomposability (aggregation consistency).

This requirement, later strengthened as an "aggregation consistency" axiom by Shorrocks (1984), says that total inequality must be the sum of a "between group" component, obtained over group **means**, and an additive component which is a weighted sum of "within group" inequalities. This kind of decomposability is very useful for controlling and dealing with heterogeneity of populations, and as a means of unambiguously identifying the sources of inequality and those that are affected by it.

In their various incarnations, Axioms 1-5 together identify the GE family as the "ideal" family of indices. But other axiom sets have been given which "justify" other inequality indices, see Blackorby and Donaldson (1978), and Dagum (1990). Indeed, one of the most popular and enduring inequality indices that is not in the GE family is the Gini index given by:

$$\begin{aligned} G &= (\frac{2}{\mu_x}) \int_0^\infty x(F - 1/2)dF \\ &= 2\int_0^\infty (F - L)dF \end{aligned} \tag{2.6}$$

where L is the Lorenz curve to be defined below. If the useful **additive** decomposability requirement of axiom 5 is imposed, such measures as Gini and variance of logarithms will be excluded. The latter two measures provide ambiguous decompositions of overall inequality by population subgroups; see Shorrocks (1984).

For the GE family a discretized (estimation) formula that helps to demonstrate its decomposability is as follows:

$$I_\gamma(X) = \sum_{r=1}^{R} [X_{.r} / \sum_i^n X_i]^{\gamma+1} (n_r/n)^{-\gamma} I^r + I_\gamma^b \tag{2.7}$$

where $X_{.r}/\sum_i X_i$ is the share of total income to group r, $r = 1, 2, ..., R$, and n_r is the number of units in that group. I^r is the "within" group GE inequality which is defined over the income shares within the $r - th$ group, and I^b is the "between" group GE inequality defined over the R group means. Shorrocks (1984) has convincingly argued that Theil's second measure ($\gamma = -1$) provides the most unambiguous answer to such fundamental questions as : How much of the overall inequality is due to the inequality in the $r - th$ group? Having a good idea about the incidence of inequality, or poverty, is an essential pre-requisite for devising well-directed and appropriate remedial action. It is also essential in establishing lower bounds for inequality that reflect acceptable differences due to heterogeneity in experience, education and skills, or other social norms. Such additive decomposability and "aggregation consistency" criterion, requiring that inequality increases if one or more I^r increase (I^b constant), are violated by Gini! We shall see further supporting arguments in favor of requiring additive decomposability in the multivariate case and in empirically relevant applications. In particular, a property of GE measures provides a partial but important degree of robustness with respect to the thorny problem of obtaining equivalence scales when we admit both the realism of heterogeneity, on the one hand, and the difficulty of correcting for it, on the other.

3. **Multivariable Welfare and Inequality**

Once the reality of heterogeneity amongst the members of a household and between households (on the basis of "needs", say) is admitted, the notion of "income inequality" itself becomes ambiguous. In a real sense the assumption of "symmetry" or "anonymity" is seen to be unacceptable. We need to adjust for heterogeneity sources before we can measure or compare "pure" income inequality.

When attributes other than money incomes are taken into account and allowed to explicitly enter the SWFs, a powerful aspect of the axioms of "fundamentality" and "anonymity" may be invoked which partially justifies common representations in preferences, in functional forms and for "representative agent" formalisms. Simply put, if we enter into the preference functions all the attributes that we think would matter in distinguishing individuals or households, then the need for ex post and often arbitrary distinctions in functional representations and other adjustments would be reduced. There are philosophical and empirical/data availability problems, but some remedies are in hand.

There are at present two inter-related lines of inquiry, or "solutions", that aim to deal with the problems of heterogeneity, equivalence scales, and what one might call a purely dimensional limitation of the univariate approach as pointed out by, for example, Kolm, Sen, Atkinson, Bourguignon, and Maasoumi..

The **first** "solution" is to search for measures that are in some sense "ideal" and preferably less sensitive to possibly incorrect methods of scaling incomes. Decomposable measures, whether of inequality or poverty, provide some protection. Additive decomposability offers an opportunity to "control" for **heterogeneity sources that are classifiable when data are collected**. Gender, age, education, income category, marital status, family size, race, ethnicity, geographic location, employment status, and many other attributes, are examples of very useful and observable characteristics which explain some sources of heterogeneity. As Coulter et al (1992) rightly argue, see also Maasoumi and Nickelsburg (1988), and Maasoumi and Zandvakili (1986, 1989, 1990), the between group component of the GE family of inequality measures is, inevitably and perhaps appropriately, the only component that is not free of how heterogeneity is defined *and/or* adjusted for. The within group components and, in certain cases, their weights are free of such "contaminations". But decomposability has its limits, both practical and because it requires comparisons of possibly many conditional inferences. Relatedly, this first approach requires cardinalization of welfare functions which involves normative comparability between individuals and households. Agreement about those normative rules that obtain unambiguous comparisons is difficult, and less stringent principles leave some ambiguity.

A **second approach** seemingly emphasizes partial orderings and eschews welfare comparisons on the basis of indices such as inequality measures. This requires stochastic dominance and other rankings of the type to be discussed below. In particular one may focus on deriving conditions for the stochastic dominance of one distribution over another on the basis of welfare functions which are, in a sense, **decomposed or separable for different population groups**. This separation both requires and opens the way for allowing different welfare evaluations for different groups which are identified by all their given characteristics other than income. Welfare comparisons of this type do not require equivalence scales, but they do require similar agreement on normative comparability of households with different characteristics and needs. This affords partial orderings but agreement on increasing levels of comparability, though difficult to reach, take us closer to cardinality and complete rankings. See our discussion of Atkinson and Bourguignon (1987, 1989) below.

It is not possible to fully avoid the difficulties mentioned above by working in terms of the univariate distribution of "equivalent income", "standard of living", or "cost of living" concepts. Some of the same normative comparability questions arise in the construction of "equivalence scales" and similar constructs. This has been brought out by Pollak and Wales (1979), Blundell and Lewbel (1991), and Shorrocks (1995). We will briefly allude to these difficulties below.

3.1. **Majority Indices.** In the multidimensional or multiattribute analysis, let X_{ij} denote a measure of attribute $j = 1, 2, ..., m$, associated with individual (unit, household, country) $i = 1, 2, ..., n$. Define the welfare matrix $X = (X_{ij})$, X_i its i-th row, X^j its j-th column, and consider any scalar function of the matrix X. Examples of such scalar functions are inequality measures or SWFs. It has proven difficult to develop "consensus" axioms which may characterize an ideal measure of multivariable inequality; more so than the univariate case discussed above. One of the main difficulties here is that, whatever the axiom sets, there is an inevitable aggregation of the m attributes that will result in any scalar measure. In view of this truism, Maasoumi (1986a) proposed a two step procedure whereby this aggregation issue is dealt with directly and explicitly. Once an "ideal" aggregation function is determined, the choice of an ideal measure of inequality may be guided by the analysis of that issue in the univariate literature. The latter analysis was sketched earlier in this paper where it identified the GE family of inequality measures.

3.1.1. *Maasoumi's two-step measures of multivariate inequality.* The aggregation of attributes in the first step has been addressed by several authors. Two broad approaches may be identified. The first is based on measures of closeness and affinity which may identify either attributes that are similar in some sense, *and/or* determine a "mean-value", or aggregate, which most closely represents the constituent attributes. The second approach which is axiomatic lays down properties that we may agree an aggregate function should possess. This second approach, recently developed by Tsui (1992b), inherits the difficulties of arriving at consensus properties which parallel the difficulty of adopting a criterion of "closeness" in the first approach. But the latter difficulty has had some resolution in "information theory" which seems to suggest members of the Generalized Entropy family as ideal criteria of "closeness" or "divergence". This topic is, however, beyond the scope of the present paper. The interested reader may see Maasoumi (1993).

Let S_i denote the aggregate or summary "well-being" function for the $i-th$ unit. I have argued elsewhere that it makes little difference to our approach whether S_i is interpreted as an individual's utility evaluations or the "observer's" or policy maker's welfare assessments for individual i. Let us define the following generalized multivariate GE measure of closeness or diversity between the m densities of m attributes:

$$D_\beta(S,X;\alpha) = \sum_{j=1}^{m} \alpha_j \{ \sum_{i=1}^{n} S_i[(S_i/X_{ij})^\beta - 1]/\beta(\beta+1)\} \tag{3.1}$$

where $\alpha_j s$ are the weights attached to each attribute. Minimizing D_β with respect to S_i such that $\sum\ S_i = 1$, produces the following "optimal" aggregation functions:

$$S_i \propto (\sum_{j}^{m} \alpha_j X_{ij}^{-\beta})^{-1/\beta}, \beta \neq 0, -1 \tag{3.2}$$

$$S_i \propto \Pi_j X_{ij}^{\alpha_j}\ ,\ \beta = 0 \tag{3.3}$$

$$S_i \propto \sum_{j} \alpha_j X_{ij}, \beta = -1 \tag{3.4}$$

These are, respectively, the hyperbolic, the generalized geometric, and the weighted means of the attributes, see Maasoumi (1986a). Noting that the "constant elasticity of substitution"-$\sigma\ =\ 1/(1{+}\beta)$, these functional solutions include many of the well known utility functions in economics, as well as some arbitrarily proposed aggregates in empirical applications. For instance, the weighted arithmetic mean subsumes a popular "composite welfare indicator" based on the principal components of X, when $\alpha_j s$ are the elements of the first eigen vector of the $X'X$ matrix; see Ram (1982) and Maasoumi (1989a).

The "divergence measure" $D_\gamma(.)$ forces a choice of an aggregate vector S = (S_1, S_2,........,S_n) with a distribution that is closest to the distributions of its constituent variables. This is desirable when the goal of our analysis is the assessment of distributional properties such as equality. Information theory establishes that any other S would be extra distortive of the objective information in the data matrix X. Elsewhere the author has argued that such a distributional criterion is desirable for justifying choices of utility, production, and cost functionals since such choices should not distort the actual market allocation signals that are in the observed data. The distribution of the data reflect the outcome of all optimal allocative decisions of all agents in the economy; see Maasoumi (1986b).

The above divergence criteria are α_j-weighted sums/averages of pairwise GE divergences between the "distributions" S and X^j, the j-th attribute/column in X. In unpublished work, the author has considered hyperbolic means of these same pairwise divergences. This generalization is capable of producing more flexible functional forms for S_i. In a different but related regression function setting originally suggested in Maasoumi (1986b advances), Ryu (1993), and Maasoumi (1993) discuss how very general flexible forms may be supported and interpreted by the Maximum Entropy (ME) method that underlies the optimization of $D_\beta(.)$ above. Finally, it is worth appreciating that the GE divergence is itself justified by a set of desirable axioms in information theory. These axioms are not very different from those in Axioms 1-5 discussed earlier; see Maasoumi (1993). This is hardly surprising since in both cases one is interested in measuring the divergence between two distributions: Indices of inequality measure the divergence between a distribution of interest and a uniform (rectangular) distribution representing perfect equality. In both cases it is found that the difference between the "entropies" of the two distributions is an ideal measure. Information theory thus provided the inspiration for both the entropy measures of Theil as well as the "axiomatic" approach of Bourguignon, Cowell and Kuga, Shorrocks, and others alluded to earlier.

The second step of the proposed method in Maasoumi (1986a) is the selection of a measure of multidimensional inequality which is the GE index of the S_i distribution just obtained. It is instructive to analyze this measure in the discrete case:

$$M_\gamma(S) = \sum_{i=1}^{n} p_i[(S_i^*/p_i)^{1+\gamma} - 1]/\gamma(1+\gamma), \gamma \neq 0, -1 \tag{3.5}$$

$$M_0(S) = \sum S_i^* \log(S_i^*/p_i), Theil's \text{ first index} \tag{3.6}$$

$$M_{-1}(S) = \sum p_i \log(p_i/S_i^*), \text{ Theil's second index} \tag{3.7}$$

where p_i is the i-th unit's population share (typically = 1/n), S_i^* is S_i divided by the total $K = \sum_{j=1}^{n} S_j$.

These inequality indices are normalized iso-elastic transformations of the utility functions $S_{i.}$ As such they are "symmetric", "homogeneous", and consistent with the Lorenz criterion. They will be homogeneous with respect to every X^j, the j-th column/attribute in X, if in all the above one works with the matrix of *shares,* $x = (x_{ij})$. While this will not change the functional solutions given above, it requires a rather unusual assertion that individual well-being depends on *shares* of attributes; see our discussion of Tsui (1992a) below.

Useful decomposability properties are possessed by these measures both in population groups and in attribute directions.

Proposition 3.1. *(Decomposability of GE)*
Let $x_{ij} = X_{ij}/T_j, T_j = \sum_i X_{ij}$, $W_j = T_j/K$, $I_\gamma(X^j) =$ *the GE inequality in the j-th attribute, and* $\delta_j = \alpha_j / \sum_{f=1}^{m} \alpha_f$. *Then:*

(i). If $1 + \gamma = -\beta$, *we have:*

$$M_\gamma(S) = \sum_{j=1}^{m} \delta_j W_j^{1+\gamma} I_\gamma(X^j) + (\sum_j \delta_j W_j^{1+\gamma} - 1)/\gamma(1+\gamma) \tag{3.8}$$

(ii). If the marginal distributions are identical-i.e., $x^j = x^k$, $\forall j$ *and* k, *we have,*

$$M_\gamma(S) = I_\gamma(X^j), any\ j \in [1, m] \tag{3.9}$$

(iii). For Theils first and second measures $(\gamma = 0, -1)$, *we have:*

$$M_0(S) = \sum_{j=1}^{m} C_j I_0(X^j) - D_{-1}(x, S^*; C) \tag{3.10}$$

where $C_j = \delta_j T_j / \sum_k \delta_k T_k$, *and,*

$$M_{-1}(S) = \sum_{j=1}^{m} \delta_j I_{-1}(X^j) - D_0(x, S^*; \delta) \tag{3.11}$$

where by application of L'Hospital's rule to $D_\beta(.)$ *defined earlier, we have:*

$$D_0(S^*, x; \delta) = \sum_j \delta_j [\sum_i S_i^* \log(S_i^*/x_{ij})] \geq 0$$
$$\textit{and } S_i \textit{ defined at } \beta = 0 \tag{3.12}$$

and,

$$D_{-1}(S^*, x; C) = \sum_j C_j [\sum_i x_{ij} \log(x_{ij}/S_i^*)] \geq 0$$
$$\textit{and } S_i \textit{ defined at } \beta = -1. \tag{3.13}$$

Proof: See Maasoumi (1986a, Propositions 1-2).

In view of the non-negativity of the D(.) terms in part (iii) of this proposition, it is clear that multidimensional inequality is no more than the weighted average of inequalities in the single attributes. This is due to the "substitution" effect between the attributes, an issue that is related to the problem of "double counting". To clarify, Maasoumi (1989b) considered the incremental contribution to inequality of a single attribute. In the special case where the attributes are jointly log-normal, the Cobb-Douglas form of the S function is also log-normal. In such cases, both of Theil's measures are given by:

$$I_0 = \frac{1}{2} C' \sum C = I_{-1} \geq 0 \tag{3.14}$$
$$= \sum_{j=1}^{m} C_j^2 (\frac{1}{2} \sigma_{jj}) + \sum_{j=1}^{m} C_j \sum_{j<k}^{m} C_k \sigma_{jk} \tag{3.15}$$

where $\sum$ is the covariance matrix of the distribution, and $C = (C_1, C_2, ..., C_m)'$ is the set of weights defined earlier, and σ_{jk} is the covariance between the (logs of) the attributes j and k. Multivariate inequality is composed of two parts. The first is a weighted sum of the attribute inequalities. The second term is an adjustment due to covariation (trade offs) between the attributes. Theil's measures are homogeneous and correspond to SWFs belonging to the class u^{--} of Atkinson and Bourguignon (1982). From the latter we learn that one multivariate distribution second order dominates another if it has lower variances and covariances. For equal marginal variances, for instance, lower (negative) covariances indicate higher multivariate inequality.

The incremental contribution, I_f say, of the f-th attribute to multivariate inequality is:

$$I_f = \frac{1}{2} C_f^2 \sigma_{ff} + C_f \sum_{k \neq f}^{m} C_k \sigma_{fk} \tag{3.16}$$
$$\geq 0 \text{ , iff } | \sum_{k \neq f}^{m} (C_k/C_f)(\sigma_{kk}/\sigma_{ff})^{\frac{1}{2}} \rho_{fk} | \leq \frac{1}{2} \tag{3.17}$$

where ρ denotes simple correlation coefficient. A sufficient condition for a positive contribution is $\sum_{k \neq f}^{m} C_f \sigma_{fk} \geq 0$. The general condition is more likely violated when attribute f is strongly *negatively* correlated with relatively more highly weighted attributes ($C_k/C_f > 1$), and or relatively less equally distributed ($\sigma_{kk}/\sigma_{ff} > 1$). Hence, negative correlation with other attributes may more than cancel the own inequality term in I_f and reduce overall inequality, see Maasoumi (1989b).

3.1.2. *Redistribution effects.* As functions of the S aggregates, SWFs corresponding to $\mathrm{M}_\gamma(.)$ indices are equality preferring since they are non-decreasing, symmetric, quasi-concave, and thus also Schur-concave. But S_i are not subject of redistributive policy, X_i, s are. The following Proposition establishes a Principle of Transfers property of the multidimensional welfare functions that is useful for redistributions of the matrix X:

Let B be a bistochastic matrix such that $\mathrm{b}_{ij} \geq 0$, $\sum_i b_{ij} = 1 = \sum_j b_{ij}\ \forall i and j$. Such matrices perform mean preserving spreads or "equalizing" transformations. Also, let H denote the set of all positive, real valued concave (concave increasing or concave non-decreasing) functions $h(.)$.

Proposition 3.2. *Let $\tilde{X} = BX$, where B is a bistochastic matrix. Then $W(\tilde{S}) \geq W(S)$ for all Schur-concave $W(.)$, and $h \in H$ such that $\tilde{S}_i = h(\tilde{X}_i)$ and $S_i = h(X_i)$.*

Proof: See Kolm (1977, Th.6). In fact the converse also holds.

Maasoumi (1986a, Proposition 3) was an attempt at deriving a similarly strong result for rankings by Schur-covex inequality measures $M_\gamma(S)$. This can be done for only a limited range of S functions, however, since M_γ does not contain all Schur-concave SWFunctions and is not everywhere increasing.

Dadanoni (1992) has considered a particular type of redistribution, a so-called "unfair redistribution" by which $\tilde{X}$ is a matrix where all the attributes are ordered in descending magnitude (The i-th individual is the i-th best off in *every* attribute). Designation of this multivariate redistribution as "unfair" implicitly assumes that all attributes are valued equally by individuals and that they are perfect substitutes! Not surprisingly, when an aversion to this type of "unfair" redistribution is made a requirement of multidimensional indices, it is found that S_i functions must be additively separable for Proposition 3 to hold *in terms of inequality indices*; see Dardanoni (1992). It may be noted that the "unfair" distribution of Dardanoni corresponds to rank dominance (majorization) in every attribute! While an aversion to this type of "unfair distribution" is desirable, it is a distribution that is as fanciful in the multidimensional context as is that of the perfectly equal reference distribution that plagues the axiomatic analysis of univariate inequality (through the assumption of *continuity*).

Our investigations show that additive separability is not required over a wide range of negative values for γ and common values of β. [3]

Finally, we note that under the conditions of Proposition 3, rankings by M_γ measures are consistent with those given by the "welfare concentration curve", a generalization of the Lorenz criterion proposed in Kolm (1977). Recently Ebert (1995b) and Shorrocks (1995) tackle a related problem of heterogenous households. Both are primarily concerned with "equivalent incomes" which can be computed if an appropriate "scale" or unit can be developed that accounts for attributes that distinguish different "needs" by members of a household. The "standard of living",

[3] A difficulty with Dardanoni's numerical examples is a failure to compute inequality indices for the size distribution of attributes (i.e., in terms of "shares"), rather than levels as is done for welfare functions. Our normalizations are such that the M_γ indices are measures of *relative* inequality (i.e., homogeneous).

or cost of living, so computed for households can replace ordinary income in an otherwise univariate inequality analysis. This is clearly the two step approach described above where the S_i are interpreted as the standard of living. Ebert (1995b) *assumes* suitable equivalence scales or standards exist! He then proposes some useful techniques for analyzing redistributions in incomes by evaluating the redistributed incomes in terms of equivalent incomes. Compared to Proposition 3 above, it is m-stochastic matrices, rather than bi-stochastic matrices, which can adequately characterize redistributions where there is substitution and complementarity between attributes. Unfortunately, Shorrocks (1995) demonstrates an “impossibility” result akin to Dardanoni’s (1992) qualification of this author’s Proposition 3 in Maasoumi (1986a). Shorrocks shows that, under apparently reasonable requirements (Equal Compensation and/or Equity Preference), unambiguous complete rankings such as those attempted in Proposition 3 above is possible only if preferences are *homothetic*. This is disappointing but not surprising. Earlier work by Pollak and Wales (1979), Pollak(1991), Lewbel (1990), and Blundell and Lewbel (1991) had uncovered this unpleasant aspect of “equivalence scales”. Identification of equivalence scales requires interpersonal welfare comparisons which are independent of a “reference” welfare level. Such independence obtains only when preferences are *homothetic*. As Blundell and Lewbel (1991) rightly point out, this is unfortunate since although it is common to adopt the assumption in theoretical analyses, there is no credible empirical support for homothetic preferences. The latest dose of evidence against homotheticity is offered by these authors using UK panel data.

Similar difficulties persist when the multivariate stochastic dominance approach is considered.

3.1.3. *The one-step measures of multivariate inequality.* Tsui (1992a, 1992b) and Ebert (1995a) have studied the possibility of a direct axiomatic derivation of multidimensional inequality measures. This approach follows the methods described earlier in relation to Axioms 1-5 of section 2. Tsui (1992a) considers SWFs that are ordinally equivalent to the additively separable individualistic SWF. Starting with the welfare matrix $X = (x_{ij})$ of n individuals (households) and m attributes, Tsui considers the following familiar axioms and derives indices of relative and “absolute” inequality which are “consistent” with these axioms:

(a) I(.) is continuous.

(b) $\mathrm{I}(X) = 0$ if X has identical rows (normalization to perfect equality).

(c) $\mathrm{I}(X) = \mathrm{I}(\Pi X)$, where Π is the permutation matrix.

This is the “anonymity” or “symmetry” assumption which is questionable in the multidimensional or heterogeneous setting.

(d) $\mathrm{I}(BX) < \mathrm{I}(X)$, where B is the bistochastic matrix defined earlier.

This is a generalization of the Lorenz dominance relations. For “relative” indices Tsui (1992a) further requires:

(e) $\mathrm{I}(XC) = \mathrm{I}\ (X)$, where $C = \mathrm{diag}(c_1,c_2,.....,c_m)$ has positive elements. This makes the measures scale invariant if and only if this property holds.

For “absolute” indices, (a)-(d) are required as well as:

(f) $\mathrm{I}(X + P) = \mathrm{I}(X)$, where P is an $n\times m$ matrix with identical rows.

Thus an index is absolute iff (f) holds.

The underlying SWFs are denoted $\mathrm{W}(.) : \mathrm{D} \rightarrow R$, where $\mathrm{D} \subseteq M$, a set of $n\times m$ matrices. They share the following properties:

(1) $\mathrm{W}(X)$ is continuous.

(2) W(X) is increasing in elements of X.

(3) W(ΠX) = W(X) where Π is the $n \times n$ permutation matrix,

(4) W(X) is strictly quasi-concave, that is W$[\alpha X+(1-\alpha)Y] > \min [W(X), W(Y)]$ for matrices $X \neq Y$, and $0 < \alpha < 1$.

Properties (3)-(4) ensure that the SWF is Lorenz consistent as in Kolm (1977) and Maasoumi (1986a, Proposition 3). That is W(BX) >W(X) for any anonymous and strictly quasi-concave W(.); see Dasgupta et al (1973).

(5) $W(X) = W(Y) \Rightarrow W(XC) = W(YC)$, for all C matrices defined earlier (relative/scale invariant), and,

(6) $W(X) = W(\varphi(X^s), X^c)$, where $\varphi(.)$ is a continuous function of X^s, a sub-matrix of X , and X^c is the complement of X^s in the partition of n individuals into s and $n-s$ groups.

Theorem 1 of Tsui (1992a) shows that properties (1)-(6) are *necessary and sufficient* to render $W(.)$ ordinally equivalent to $\sum_i U(X^i)$, where X^i is the i-th row of X, and $U^i(.)$ is the i-th individual's strictly increasing and concave valuation function taking the following forms:

$$a + b\Pi_{j=1}^m x_j^{r_j} \tag{3.18}$$

or,

$$a + \sum_{j=1}^m r_j \log x_j \tag{3.19}$$

where a is an arbitrary constant, and b and r_j are restricted so that $U(.)$ is increasing and strictly concave.

The corresponding relative inequality index is :

$$M_r = 1 - [\frac{1}{n}\sum_{i=1}^n \Pi_{j=1}^m (\frac{x_{ij}}{\mu_j})^{r_j}]^{\frac{1}{\sum r_j}} \tag{3.20}$$

and,

$$M_r = 1 - \Pi_{i=1}^n [\Pi_{j=1}^m (\frac{x_{ij}}{\mu_j})^{\frac{r_j}{\sum_k r_k}}]^{\frac{1}{n}} \tag{3.21}$$

where μ_j is the mean of the j-th attribute. These are clearly the multivariate generalizations of Kolm's (1977) univariate indices of inequality. They resemble some members of the class $M_\gamma(x)$ proposed by the present author.

Tsui (1992a) further considers absolute inequality measures. Replacing Property (5) with

(7) $W(X) = W(Y) \Rightarrow W(X+P) = W(Y+P)$, where P has identical rows, Tsui demonstrates the following result:

(i) $W(X)$ satisfies Properties (1)-(4) and (6)-(7) if and only if $U(.)$ is strictly increasing and concave, taking the following forms:

$$U(X) = b\Pi_{j=1}^m \exp(r_j x_j) + a \tag{3.22}$$

where a is arbitrary and b and r_j should respect the properties of $U(.)$.

(ii) The corresponding functional form for the absolute inequality measures is:

$$M_A = \log\left\{\frac{1}{n}\sum_{i=1}^{n}\exp\left(\sum_{j=1}^{m} r_j(\mu_j - x_{ij})\right)\right\} \tag{3.23}$$

Tsui (1992a) further explicates the constraints on the free parameters which restrict their signs and impose restrictions on their values to insure the properties of $U(.)$ are maintained.

Ebert (1995a) has also studied the modification of Axioms 1-5 in the two-dimensional case of incomes and "household types". Following Atkinson and Bourguignon (1987), he considers the situation in which household types are ordinal and discrete. Then grouping households by type, the following axioms are postulated:

A1. "partial symmetry", means that different household types can be treated asymmetrically but the previous anonymity axiom is adopted for members in a category.

A2. "Aggregation" . Suppose there are two populations. The first is indifferent between two income situations $X^{(1)}$ and $Y^{(1)}$, and the second population is indifferent between $X^{(2)}$ and $Y^{(2)}$. Then this axiom requires that the combined populations should be indifferent between ($X^{(1)}$,$X^{(2)}$) and ($Y^{(1)}$,$Y^{(2)}$). This axiom reduces to the aggregation consistency axiom in the univariate case.

A3. Generalized Pigou-Dalton Principle (equality preference).

This requires a known ordering of household types by "needs", say. Then "progressive" income transfers that don't change the income ranks of individuals *after adjustment for their different needs,* are preferred by the SWF.

A4. Generalized scale invariance. As in Tsui (1992a) described earlier.

Imposing these properties on SWFs, and like Tsui (1992a), using the Kolm-Atkinson measure of inequality based on "equivalent income" described earlier, produces a class of measures which, as Ebert shows, are ordinally equivalent to the present author's M_γ measures above.

Comparison of welfare matrices forces interpersonal comparisons of well-being which, in turn, require further cardinal statements regarding the trade offs between different attributes by different households. This means that "consensus" measures are impossible to obtain. *Majority* indices are possible, however, since the indices such as those proposed by this author are shown to be consistent with many plausible axioms which command wide-spread agreement. This situation is the same even for partial orderings to be discussed under stochastic dominance. One must persist in this direction, however, as the need for measurement endures, and since it seems difficult to define significant social values that have no individualistic impact and are therefore not capable of evaluation at the level of individuals (welfare units)[4]. Resolution surely lies with abandoning a mode of thinking about attributes as separate, qualitatively "equal", and either perfectly substitutable or not at all. The most plausible situations are those where having a little of some attribute is compensated by having more of another. At this level, if heterogeneity in tastes cannot be ordered, the hope for *consensus*, individualistic, multidimensional measurement would be dimmed.

[4]The evaluation does not need to be conducted by the welfare unit itself. As I discussed in Maasoumi (1986a), these evaluations may be conducted by an observer. Shorrocks (1996) also suggests that analyses proceed by thinking in terms of these living standards/aggregates. All the existing results and norms clearly apply to ranking of the resulting "univariate" distributions.

A related issue of "double counting" latent attributes was anticipated by Hirschberg, Maasoumi, and Slottje (1991). These authors explored the properties of "cluster analysis" as a means of reducing dimensionality in the current context. The idea is to identify clusters of statistically "similar" attributes which may then be represented by a single indicator. Further description is given in a later section.

3.2. **Stochatic Dominance.** The second approach alluded to earlier is based on the desire to avoid full cardinalization that is required for index choice. We first describe the elements of ordering and statistically testing for stochastic dominance in the univariate case. The definitions and modified forms of the tests carry over to the multidimensional case which will be discussed in a subsequent section.

3.2.1. *Definitions and Tests in the Univariate Case.* Let X and Y be two income variables at either two different points in time, before and after taxes, or for different regions or countries. Let $X_1, X_2, ..., X_n$be n not necessarily i.i.d observations on X, and $Y_1, Y_2, ..., Y_m$ be similar observations on Y. Let U_1 denote the class of all utility functions u such that u' ≥ 0, (increasing). Also, let U_2 denote the class of all utility functions in U_1 for which $u'' \leq 0$ (strict concavity), and U_3 denote the subset of U_2 for which $u''' \geq 0$. Let $X_{(i)}$ and $Y_{(i)}$ denote the i-th order statistics, and assume F(x) and G(x) are continuous and monotonic cumulative distribution functions (cdf,s) of X and Y, respectively.

Quantiles $q_x(p)$and $q_y(p)$ are implicitly defined by, for example, $F[X \leq q_x(p)] = p$.

Definition 3.1. *X First Order Stochastic Dominates Y, denoted X FSD Y, if and only if any one of the following equivalent conditions holds:*

(1) E[u(X)] $\geq$ *E[u(Y)]* *for all* $u \in U_1$, *with strict inequality for some u.*

(2) F(x) $\leq$ *G(x)* *for all x in the support of X, with strict inequality for some x.*

(3) $q_x(p) \geq q_y(p)$ *for all* $0 \leq p \leq 1$.

Definition 3.2. *X Second Order Stochastic Dominates Y, denoted X SSD Y, if and only if any of the following equivalent conditions holds:*

(1) E[u(X)] $\geq$ *E[u(Y)]* *for all* $u \in U_2$, *with strict inequality for some u.*

(2) $\int_{-\infty}^{x} F(t)dt \leq \int_{-\infty}^{x} G(t)dt$ *for all x in the support of X and Y, with strict inequality for some x.*

(3) $\int_0^p q_x(t)dt \geq \int_0^p q_y(t)dt$ *, for all* $0 \leq p \leq 1$ *, with strict inequality for some value(s) p.*

The tests of FSD and SSD are based on empirical evaluations of conditions (2) or (3) in the above definitions. Mounting tests on conditions (3) typically relies on the fact that quantiles are consistently estimated by the corresponding order statistics at a finite number of sample points. Mounting tests on conditions (2) requires empirical cdfs and comparisons at a finite number of observed ordinates. Also, from Shorrocks (1983) or Xu (1995) it is clear that condition (3) of SSD is equivalent to the requirement of Generalized Lorenz (GL) dominance. FSD implies SSD.

Noting the usual definition of the Lorenz curve of, for instance, X as $L_x(x) = \frac{1}{\mu_x} \int_{-\infty}^{x} X \times dF(t)$, and its GL (x) = $\mu_x L_x$ (x) , some authors have developed tests for Lorenz and GL dominance on the basis of the sample estimates of conditional interval means and cumulative moments of income distributions; e.g. see Bishop et al (1989), Bishop et al (1991), Beach et al (1995), and Maasoumi (1996a) for a general survey of the same. The asymptotic distributions given by Beach et al (1995) are particularly relevant for testing for Third Order Stochastic Dominance (TSD). The latter is a useful criterion when Lorenz or GL curves cross at several points and the investigator is willing to adopt "transfer sensitivity" of Shorrocks and Foster (1987), that is a relative preference for progressive transfers to poorer individuals. When either Lorenz or Generalized Lorenz Curves of two distributions cross unambiguous ranking by FSD and SSD is not possible. Whitmore (1970) introduced the concept of third order stochastic dominance (TSD) in finance. Shorrocks and Foster (1987) showed that the addition of the "transfer sensitivity" requirement leads to TSD ranking of income distributions. This requirement is stronger than the Pigou-Dalton principle of transfers and is based on the class of welfare functions U_3. TSD is defined as follows:

Definition 3.3. *X Third Order Stochatic Dominates Y, denoted X TSD Y, if and only if any of the following equivalent conditions holds:*

(1) E[u(X)] $\geq$ E[u(Y)]   for all u $\in U_3$, with strict inequality for some u.

(2) $\int_{-\infty}^{x}\int_{-\infty}^{v}[F(t)-G(t)]dt\,dv \leq 0$, for all x in the support, with strict inequality for some x,

with the end-point condition:

$\int_{-\infty}^{+\infty}[F(t)-G(t)]dt \leq 0.$

(3) When E[X] = E[Y], X TSD Y iff $\lambda_x^2(q_i) \leq \lambda_y^2(q_i)$, for all Lorenz curve crossing points $i = 1, 2, \ldots\ldots.$, $(n+1)$; where $\lambda_x^2(q_i)$ denotes the "cumulative variance" for incomes upto the *ith* crossing point. See Davies and Hoy (1996).

When $n = 1$, Shorrocks and Foster (1987) showed that X TSD Y if (a) the Lorenz curve of X cuts that of Y from above, and (b) Var(X) $\leq$ Var(Y). This situation revives the coefficient of variation and covariances as useful statistical indices for ranking distributions.

Recently, Kaur et al (1994) proposed a test for condition (2) of SSD when i.i.d observations are assumed for independent prospects X and Y. Their null hypothesis is condition (2) of SSD *for each x* against the alternative of strict violation of the same condition *for all x.* The test of SSD then requires an appeal to union intersection technique which results in a test procedure with maximum asymptotic size of α if the test statistic at each x is compared with the critical value Z_α of the standard Normal distribution.

In the area of income distributions and tax analysis, initial developments focusing on statistical tests for Lorenz curve comparisons are exemplified by Beach and Davidson (1983), Bishop, Formby, and Thistle (1989), and Bishop et al (1991) . In practice, a finite number of ordinates of the desired curves or functions are compared. These ordinates are typically represented by quantiles and/or conditional interval means. Thus, the distribution theory of the proposed tests are typically derived from the existing asymptotic theory for ordered statistics and quantiles. RecentlyBeach, Davidson, and Slotve (1995) have outlined the asymptotic distribution theory for cumulative/conditional means and variances which are the essential ingredients of Lorenz and GL curves. To control for the size of a sequence of tests at several points the Union Intersection (UI) test and Studentized Maximum Modulus technique for multiple comparisons is generally favored in this area.

More recently several non-parametric tests have been proposed for FSD and SSD which recognize that distribution functions are unknown and have to be empirically estimated. The McFadden and Kaur et al tests are in this spirit. Some alternatives to these multiple comparison techniques have been suggested which are typically based on Wald type joint tests of equality of the same ordinates, see Bishop et al (1994) and Anderson (1994). These alternatives can be somewhat problematic when their implicit null and alternative hypotheses fail to clearly deal with the **inequality** (order) relations that need to be tested. Xu et al (1995), and Xu (1995) take proper account of the inequality nature of such hypotheses and adapt econometric tests for inequality restrictions to testing for FSD and SSD, and to GL dominance, respectively. Their tests follow the work in econometrics of Gourieroux et al (1982) Kodde and Palm (1986), and Wolak (1988, 1989), which complements the work in statistics exemplified by Kudô (1963), Perlman (1969), Robertson and Wright (1981), and Shapiro (1985). Good general accounts are given in Robertson et al (1981) and Shapiro (1988). The asymptotic distributions of these $\chi - bar$ squared tests are mixtures of chi-squared variates with probability weights which are generally difficult to compute. This leads to the suggestion of bounds tests involving inconclusive regions and conservative inferences. In addition, the computation of the χ *bar* squared statistic requires Monte Carlo or Bootstrap estimates of covariance matrices, as well as inequality restricted estimation which requires optimization with quadratic linear programming. In contrast, Maasoumi et al (1996) propose a direct bootstrap approach that bypasses many of these complexities while making less restrictive assumptions about the underlying processes.

McFadden (1989) and Klecan, McFadden, and McFadden (1991) have proposed tests of first and second order "maximality" for stochastic dominance which are extensions of the Kolmogorov-Smirnov (KS) statistic. McFadden (1989) assumes i.i.d. observations and independent variates, allowing him to derive the asymptotic distribution of his test, in general, and its exact distribution in some cases (see Durbin (1973, 1985). He provides a Fortran and a GAUSS program for computing these tests. Klecan et al generalize this earlier test by allowing for weak dependence in the processes both across variables and observations. They demonstrate with an application for ranking investment portfolios. The asymptotic distribution of these tests cannot be fully characterized, however, prompting Klecan et al to propose Monte Carlo methods for evaluating critical levels. Maasoumi et al (1996) develop a testing procedure based on bootstrap estimated confidence intervals for the KS statistic . In the following subsections some definitions and results are summarized which help to describe our tests. The McFadden-type tests require a definition of "maximal" sets, as follows:

Definition 3.4. *Let $= \{X_1, X_2, ..., X_K\}$ denote a set of K distinct random variables. Let F_k denote the cdf of the kth variable. The set Æ is first (second) order maximal if no variable in Æ is first (second) order weakly dominated by another.*

Let $X_{.n} = (x_{1n}, x_{2n}, ..., x_{Kn})$, n = 1, 2,....,N , be the observed data. We assume $X_{.n}$ is strictly stationary and $\alpha - mixing$. As in Klecan et al., we also assume $F_i(X_i)$, i = 1,2,...,K are *exchangeable* random variables, so that our resampling estimates of the test statistics converge appropriately. This is less demanding than the assumption of independence which is not realistic in many applications (as in before and after tax scenarios). We also assume F_k is unknown and estimated by the empirical distribution function $F_{kN}(X_k)$. Finally, we adopt Klecan et al's mathematical regularity conditions pertaining to von Neumann-Morgenstern (VNM) utility functions that generally underlie the expected utility maximization paradigm. The following theorem defines our tests and the hypotheses being tested:

Theorem 3.3. *Given the mathematical regularity conditions;*

(a) The variables in Æ are first-order stochastically maximal; i.e.,

(1) $d = \min_{i \neq j} \max_{x} [F_i(x) - F_j(x)] > 0,$ (1)

if and only if for each i and j, there exists a continuous increasing function u such that $E\, u(X_i) > E\, u(X_j)$.

(b) The variables in Æ are second order stochastically maximal; i.e.,

(1) $S = min_{i \neq j}\ max_x \int_{-\infty}^{x} [F_i(\mu) - F_j(\mu)] d\mu > 0,$ (2)

if and only if for each i and j, there exists a continuous increasing and strictly concave function u such that $Eu(X_i) > Eu(X_j)$.

(c) Assuming the stochastic process $X_{\cdot n}$ *, n = 1, 2,.... , to be strictly stationary and* $\alpha - mixing$ *with* $\alpha(j) = O(j^{-\delta})$, *for some* $\delta > 1$, *we have:*

$d_{2N} \rightarrow d$ *, and* $S_{2N} \rightarrow S$, *where* d_{2N} *and* S_{2N} *are the empirical test statistics defined as :*

(1) $d_{2N} = \min_{i \neq j} \max_{x} [F_{iN}(x) - F_{jN}(x)]$ (3)

and,

$S_{2N} = \min_{i \neq j} \max_{x} \int_0^x [F_{iN}(\mu) - F_{jN}(\mu)] d\mu$ (4)

Proof. . See Theorems 1. and 5 of Klecan et al (1991).

The null hypotheses tested by these two statistics is that, respectively, Æ is *not* first (second) order maximal— i.e., X_i FSD(SSD) X_j for some i and j. We reject the null when the statistics are positive and large. Since the null hypothesis in each case is composite, power is conventionally determined in the least favorable case of identical marginals $F_i = F_j$. Thus, as is shown in Kaur et al (1994) and Klecan et al (1991), tests based on d_{2N} and S_{2N} are consistent. Furthermore, the asymptotic distribution of these statistics are non-degenerate in the least favorable case, being Gaussian (see Klecan et al (1991), Theorems 6-7).

As is pointed out by Klecan et al (1991), the statistic S_{2N} has, in general, neither a tractable distribution, nor an asymptotic distribution for which there are convenient computational approximations. The situation for d_{2N} is similar except for some special cases—-see Durbin (1973, 1985), and McFadden (1989) who assume i.i.d. observations (not crucial), and independent variables in Æ (consequential). Unequal sample sizes may be handled as in Kaur et al (1994).

Klecan et al (1991) suggest Monte Carlo procedures for computing the significance levels of these tests. This forces a dependence on an assumed parametric distribution for generating MC iterations, but is otherwise quite appealing for very large iterations. The bootstrap method proposed by Maasoumi et al (1996) is distribution-free and quite accurate. Pilot studies show that our computations obtain similar results to the algorithm proposed in Klecan et al for Normal distributions.[5]

[5] Our program is written in GAUSS and is available upon request.

In their algorithm Maasoumi et al (1996) compute d_{2N} and S_{2N} for a finite number K of the income ordinates. This requires a computation of sample frequencies, cdfs and sums of cdfs, as well as the differences of the last two quantities at all the K points. Next, bootstrap samples (typically 1000) are generated from which empirical distributions of the differences and of the d_{2N} and S_{2N} statistics are determined. Confidence intervals are obtained which allow statistical rejection (non-rejection) of dominance. An example based on the CPS data, and another based on the PSID are given in the last section of this paper.

3.2.2. *Multivariate Stochastic Dominance.* Atkinson and Bourguignon (1982, 1987, 1989) have developed the conditions for ranking multidimensioned distributions of welfare attributes. SWFs are taken to be individualistic and (for convenience) separable. But anonymity may be dropped in recognition of the fact that households (individuals) must be distinguished according to their distinct needs or other characteristics. It is desired to rank the distributions of incomes in two states conditional on a given distribution of discrete characteristics such as needs or family composition.

Let there be G groups which are characterized in terms of their "needs" and incomes. All members of a group g $\in$ G have the same valuations and marginal valuations of income. If there were no income transfers between groups the necessary and sufficient conditions for FSD and SSD given above must hold for *all* groups for FSD and SSD to hold. If there is any transfer between groups, however, one must deal with each group's evaluation as well as between group valuations of the trade-offs between income and "needs". Thus interpersonal comparisons of well-being are inevitable whenever heterogenous populations are involved. It is here that Shorrocks' (1996) results are the latest demonstration of an "impossibility" of unambiguous or consensus rankings. But, "majority" rankings are possible with plausible restrictions. To see this it is worthwhile to formally describe the conditions of Atkinson and Bourguignon (1987) here as they combine the desirable elements of "decomposability" alluded to at the beginning of this section, and partial ordering which, although it avoids full cardinalization, shows the directions in which an analyst may wish to make increasingly normative assumptions to *approach cardinality*; see Basu (1980).

Let u(X,h) denote private valuations of income X and "handicaps" or needs h, w(u(X,h)) or just w(X,h) represent the social welfare (or decision) function, and p_g, g=1,2,....,G, the marginal frequency in class g. The cumulative function is $P_g = \sum_{j=1}^{g} p_j$, P_G =1. The social valuation of income received by household g is $U^g(X)$ which is assumed continuously differentiable as needed. The U_1 and U_2 classes are as defined earlier with the partial derivatives $U_X^g \geq 0$, as well as $U_{XX}^g \leq 0$ for U_2.

If no assumptions are made about how U^g varies with g the conditions of FSD and SSD must hold for all groups g for FSD and SSD to hold. These are strong conditions. Among other things, they require that the mean income of all groups must be no lower in the dominant distribution. This would rule out equalizing redistributions between groups with different needs. To resolve this situation one must specify some aspects of the trade-off between incomes and needs.

The traditional univariate/homogenous analysis is implicitly based on the extreme assumption that $U_X^g(X) = U_X(X)$, $\forall g$. The *level* of welfare can vary with needs, but no more. This assumption is sufficient to allow a consideration only of the marginal distribution of income. But suppose one follows Sen in weakening the

"Equity Axiom" by assuming that groups can be ranked by their *marginal valuation of income.* For instance, if the neediest group has the highest marginal valuation of income, the next neediest group has the second highest marginal valuation, and so on, then the necessary and sufficient condition for FSD of F_1 over F_2 is:

$$\sum_{g=1}^{j} p_g[F_1^g - F_2^g] \leq 0, forallXandallj = 1, ..., G \tag{3.24}$$

where superscript indicates the income distribution for g-th group. Note that the final condition here is the FSD of the entire marginal distribution of incomes. As has shown by Sen (1973b), see also Arrow (1971), it is possible for a utilitarian SWF to violate this type of "weak equity axiom". But as Atkinson and Bourguignon (1987) point out, marginal valuation by *society* can take into account the *level* of individual welfare. Therefore it is possible that the assumed negativity of u_{Xh} may be offset by sufficiently degree of concavity ($-w'' / w'$) of the additive social valuation function w(.). Thus the ranking of groups assumed by Atkinson and Bourguignon (1987) coincides with the ranking of levels of welfare where higher needs increase marginal valuations of income, or the social welfare function has a sufficiently large degree of concavity.

The above FSD condition may be weakened further for SSD if we are willing to assume that "the differences in the social marginal valuation of income between groups become smaller as we move to higher income levels"; se Atkinson and Bourguignon (1987). That is -U_{XX} decreases for less needy groups reflecting less social concern with differences in needs for higher income groups. If this assumption is adopted a necessary and sufficient condition for SSD is:

$$\sum_{g=1}^{j} p_g[\int_0^x (F_1^g - F_2^g)dX] \leq 0 \text{ for all } x, \text{ and } j = 1, ..., G \tag{3.25}$$

this includes the usual SSD condition for the marginal distribution of incomes.

Atkinson and Bourguignon (1987) consider weaker SSD conditions by exploring further assumptions toward cardinality. One such assumption allows further comparability between the *differences* of U_X and U_{XX}. Thus, if the rate of decline of social marginal valuation of income across groups is positive *and declines with* g, *and* the same property holds for the degree of concavity ($-U_{XX}$), the necessary and sufficient condition for SSD is given as follows:

$$\sum_{j=1}^{k} \sum_{g=1}^{j} p_g[\int^x (F_1^g - F_2^g)dX] \leq 0 \text{ for all } x \text{ and k=1,....,G-1}$$

$$and$$

$$\sum_{g=1}^{G} p_g[\int^x (F_1^g - F_2^g)dX] \leq 0 \text{ for all } x \tag{3.26}$$

It is worth noting that all the above conditions are testable using the tests outlined above.

Consistent with a philosophy of "partial comparability" developed by Sen (1970b), Atkinson and Bourguignon (1987) have therefore shown that nihilism is avoidable if certain plausible assumptions are made about the trade offs between incomes and needs, and at different levels of needs, should we agree that groups can be ranked by such "other" characteristics as "needs". These additional assumptions lead to the development of empirically implementable tests for stochastic dominance which are somewhat more general but less demanding than those described earlier.

4. Further Statistical Tools for Inference

A more detailed account of the available statistical tools for inference about indices is given in Maasoumi (1996a). Here a brief account is given that exemplifies the range of available techniques.

We first consider the direct MM estimation of inequality measures and their standard errors which permit construction of asymptotic confidence intervals. Of course, the most general treatment will have to be in terms of equivalence scale-adjusted incomes for individuals. Thus, following Cowell (1989), if the total household income is $y(i)$, its characteristics vector is c_i, and the "adult equivalent" function is $\zeta_i = \zeta(y(i), c_i)$, then the adjusted "income" variable is $x_i = y(i)/\zeta_i$ with corresponding weights ζ_i. We deal with the simplest example of ζ, *i.e.*, household size h_i. Then we may write the GE family of inequality indices as follows :

$$I_\gamma = [\mu_{1\gamma}\mu_{11}^{-(\gamma+1)}\mu_{10}^{\gamma} - 1]/\gamma(\gamma+1), \gamma \neq -1, 0 \tag{4.1}$$

μ_{ij} are the raw moments of the joint distribution of household size H and "income" X; for instance

$$\mu_{i\alpha} = \int\int h^i x^\alpha dF(h,x), i = 1,2, -\infty < \alpha < \infty \tag{4.2}$$

The specialization to Theil's two measures are:

$$I_0 = \log \mu_{11} - \log \mu_{10} - \tau_{10}/\mu_{10}$$

$$I_{-1} = \tau_{11}/\mu_{11} - \log\ \mu_{11} + \log\ \mu_{10} \tag{4.3}$$

where,

$$\tau_{ij} = \int\int h^i x^j \log\ x\ dF(h,x)\ ,\ i,j = 0,1,2 \tag{4.4}$$

MM estimators of I_γ are obtained by replacing the population moments with their sample counterparts. For example, $\mu_{i\alpha}$ is estimated by:

$$m_{i\alpha} = \sum_{l}^{n} h_l^i x_l^\alpha / n, fornhouseholds, \tag{4.5}$$

Since inequality measures are functions of population moments, the usual techniques such as the delta method may be used to approximate variances and asymptotic distributions. For example, if the vectors of the population and sample moments are denoted by, respectively μ and m, we let $I_\gamma = g(\mu)$ and its MM estimator, $\hat{I}_\gamma = g(m)$. Under regularity conditions (certainly with random sampling or certain forms of stratified sampling), $\sqrt{n}(m-\mu)$ is asymptotically normal with zero mean and covariance matrix $\sum$. And if $g(.)$ is differentiable, we may base inferences on the following asymptotic distribution:

Theorem 1:

$$\sqrt{n}(\hat{I}_\gamma - I_\gamma) \sim N(0, V) \tag{4.6}$$

where the asymptotic variance matrix is computed from:

$$V = G' \sum G \ /n \tag{4.7}$$

and the elements of $G = \partial g/\partial \mu$ are obtained as follows:

$$\partial g/\partial \mu_{1\gamma} = \varphi_\gamma/\mu_{1\gamma}, \gamma \neq 0, -1 \tag{4.8}$$

$$\delta g/\delta \mu_{11} = -(\gamma + 1)\varphi_\gamma/\mu_{11}, \gamma \neq 0 \tag{4.9}$$

$$= -\varphi_1/\mu_{11}, \gamma = 0 \tag{4.10}$$

$$\partial g/\partial \mu_{10} = \gamma\varphi_\gamma/\mu_{10}, \gamma \neq -1 \tag{4.11}$$

$$= \varphi_0/\mu_{10}, \gamma = -1 \tag{4.12}$$

where,

$$\varphi_\gamma = \mu_{1\gamma}\mu_{11}^{-(\gamma+1)}\mu_{10}^{\gamma}/\gamma(\gamma + 1), \gamma \neq 0, -1, and, \varphi_i = \tau_{1i}/\mu_{1i} + (2i - 1), i = 0, 1. \tag{4.13}$$

Cowell (1989) derived the expressions for an estimator of V. These were corrected by Mills and Zandvakili (1996) who propose bootstrap estimation for the same. Maasoumi (1996a) contains a survey of statistical distribution theory in this area that covers other indices, including the Gini coefficient, and quantile-based statistics which are often needed for measuring Lorenz-type curves and other ranking relations.

5. Empirical experience

The number of multiattribute applications has grown steadily. US and UK data, international data on GDP, Basic Needs, and Physical Quality of Life Indicators (PQLI) data, and some other "country studies" have used both the Maasoumi indices and the FSD and SSD tests. We will briefly reference some of these applications here.

It is worthwhile to first note a possible difficulty with potential "double counting" of the same *latent* welfare components by inclusion of measurements on *apparently* distinct attributes. Put differently, two apparently distinct attributes may offer almost identical amounts of "information" to the information set inevitably utilized by any statistical measure. This issue is studied by Hirschberg, Maasoumi, and Slottje (1991) for international data. The basic idea is to detect "clusters" of attributes which are statistically similar. Once this is accomplished, a "representative" aggregator attribute is chosen for each cluster. These representative or composite attributes are then included in the desired but lower dimensional multivariate welfare analysis. The approach of Hirschberg et al (1991) is based on statistical clustering techniques as well as a new entropy based criterion. In Hirschberg et al. (1991) 24 attributes of well being were analyzed for 120 countries. These included such attributes as GNP and related concepts, literacy and mortality rates, labor force participation rates, basic amenities (*e.g.*, radios and roads), militarization indices, health status, infrastructure indicators, political freedom and civil liberty measurements. Interesting and quite plausible “clusters” were identified based on several criteria of similarity. The authors then proceeded to compute aggregate measures of well being on the basis of the “representative” attributes for the five clusters, and computed Maasoumi’s measures of multidimensional inequality. This type of study also allows an investigation of the robustness of inferences, for example, with respect to levels of aggregation (clustering), weighting factors, and aversion to inequality (parameter γ of GE).

Maasoumi and Jeong (1985) consider composite measures of well-being for 120 countries based on PQLI, Basic Needs indicators, and GDP per capita. Their aggregates include composite measures earlier suggested by Ram (1982) for the same data. Maasoumi and Jeong (1985) report both decompositions of multivariate inequality by attribute and by country characteristics such as degree of industrialization, market orientation, and geographical location. They find, for instance, that a good deal of income inequality existed amongst the “centralized” economies which was greatly reduced in the multidimensional measures by their very equal distributions in quality of life indicators. This finding was fairly robust with respect to the relative weights given to the different attributes. It is possible, however, to consider extreme degrees of inequality aversion (very high values of γ) which would respond greatly to a very low level in some attribute. It is debatable whether such a phenomenon is undesirable when all the included attributes are presumably considered worthy of inclusion, and a degree of inequality aversion is agreed upon on ethical grounds.

Maasoumi and Nickelsburg (1988) study randomized samples of the PSID households in a multidimensioned analysis of inequality based on such diverse attributes as education (years of schooling), total real incomes, and “wealth” proxied by net equity in housing.

The type of study just mentioned highlights several main difficulties. Firstly, data availability and reliability, especially for international comparisons . Secondly, a classic problem of index numbers exacerbated by the unavailability of observable behavioral (structural) relations. Not all or even most attributes are traded or tradable. Thus, the usual econometric approaches to ordinary commodities, whatever their success elsewhere, are generally not available in this context. Finaly, a somewhat subtler problem is that, while indexing techniques and inequality measures are based on sometimes elaborate distance metrics, a simple “eye-balling”, linear metric is used to compare index values for different welfare units (countries, say) or at different points. The similarity or “divergence” between units, and intertemporal and spacial considerations are thus analyzed primitively.

In a series of papers Maasoumi and Zandvakili (1986,1989, 1990) treat income at different points in time as distinct attributes. This is an interesting application of multidimensional analysis to the analysis of “inequality reducing” mobility measures which allows for more realistic and sounder aggregation of income over increasingly longer periods of time. First “permanent” income over increasing intervals over the life-cycle are constructed using the entropy based aggregator functions described above. These trivially include the simple sum of incomes over time which is restrictive with respect to intertemporal substitution. Then the M_γ measures are applied to analyze two issues. One is the question of “long run” or permanent income inequality thus made relatively free of transitory effects. The second issue is a consideration of inequality reducing mobility measures first introduced by Shorrocks (1978). The generalized income measures lead to a generalization of Shorrocks' mobility indices.

Maasoumi and Zandvakili study the data from the Panel Study of Income Dynamics (PSID) allowing decompositions by age, gender, education, and race. The authors confirm the moderating influence on inequality of looking at "long run income" measured over the life cycle. They also find a robust pattern in the mobility profiles which suggests improvements of early 70s were lost by the 80s. The topic of mobility is the subject of a survey in Maasoumi (1996b).

Panel data studies, such as Shorrocks (1978), Maasoumi and Nickelsburg (1988), and Maasoumi and Zandvakili (1986, 89, 90), demonstrate the great advantage of such data which, by virtue of recording detailed characteristics of households over time, afford the analyst the opportunity to control and adjust for characteristics that represent heterogeneity amongst groups. As we have seen, this may prove an essential element in the multidimensional context which could permit homogeneous treatment of within group units, and a possible homogenization of the "distances" between groups that render more plausible the type of "comparability" assumptions made by Atkinson and Bourguignon (1987) and Shorrocks (1995). This is not an area of analysis that could be furthered by the usual grouped data reporting group sizes and mean incomes. It would be missing the point to think that the annual reports about which groups earned which percentage of incomes have the meaning that they seemingly impart. The common apathy and cynicism that is observed relative to distributive issues, both in the population at large and amongst large numbers of academic economists, is very much a result of an instinctive belief that income disparity between patently distinct households has unclear ethical ramifications. Also, there are people of great wealth and comfort (non-income rich) whose well being cannot possibly be measured with their reported income. There are clear lessons about the type of data that needs to be collected. The UK and other panel studies (such as for the European Common Market in Luxemburg) are expensive but welcome developments in this direction. International comparisons have also been enhanced by the type of data compiled by the World Bank which Maasoumi and Jeong (1985), Ram (1982), and others have analyzed.

A point concerning measurement error, particularly with international data, is worth emphasizing. Accurate data are as important to science as the method of science. But there is little to support the notion that the missing parts of the supposedly accurate national income accounts, due to the operation of the "under-ground economy", say, are relatively smaller than the potential classification errors made in categorical data regarding "needs", degree of "civil liberty", or Basic Needs indicators such as calorie intake and access to medical amenities. Indeed, categorization of data is a potentially profitable means of reducing relative errors in data collection and variable definitions.

Concerning the theoretical obstacles cited earlier in the way of "consensus" rankings of multidimensioned welfare situations, it is worth noting that scientific decisions do not need to be unanimous. They need to be plausible and replicable. The interpretation of empirical results under the "majority" axioms described here is unambiguous. In this respect, there is an empirical method of robustification that plays an important role. It consists of empirically identifying qualitative inferences which do not change significantly as we vary the weights given to different attributes, and patterns of distributive change that are robust to variation over reasonable degrees of aversion to inequality and/or poverty. This is important for reliability, but also because all inferences are statistical when the true population is not known.

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