

Section 7.6: Inverse Trigonometric Functions

Inverse Sine Function:

The inverse function of $\sin x$ is written as $\sin^{-1} x$ or as $\arcsin x$. The function $\sin^{-1} x$ gives the value whose sin is x . We define it formally by:

$$y = \sin^{-1} x \Leftrightarrow \sin y = x \quad \text{and} \quad y \in [-\pi/2, \pi/2]$$

Cancellation Equations:

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\pi/2 \leq x \leq \pi/2$$

$$\sin(\sin^{-1} x) = x \quad \text{for} \quad -1 \leq x \leq 1$$

Inverse Cosine Function:

The inverse cosine function is defined similarly:

$$y = \cos^{-1} x \Leftrightarrow \cos y = x \quad \text{and} \quad y \in [0, \pi]$$

Cancellation Equations:

$$\cos^{-1}(\cos x) = x \quad \text{for} \quad 0 \leq x \leq \pi$$

$$\cos(\cos^{-1} x) = x \quad \text{for} \quad -1 \leq x \leq 1$$

Inverse Tangent Function:

The inverse tangent function is defined by:

$$y = \tan^{-1} x \Leftrightarrow \tan y = x \quad \text{and} \quad y \in (-\pi/2, \pi/2)$$

The following limits of $\tan^{-1} x$ hold:

$$\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Other Inverse Trigonometric Functions:

The remaining inverse trig functions are defined as follows:

$$y = \csc^{-1} x (|x| \geq 1) \Leftrightarrow \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1} x (|x| \geq 1) \Leftrightarrow \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$y = \cot^{-1} x (x \in \mathbb{R}) \Leftrightarrow \cot y = x \quad \text{and} \quad y \in (0, \pi)$$

Derivatives of Inverse Trigonometric Functions:

$$\begin{aligned}\frac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}(\csc^{-1} x) &= -\frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}(\cos^{-1} x) &= -\frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}(\sec^{-1} x) &= \frac{1}{x\sqrt{x^2-1}} \\ \frac{d}{dx}(\tan^{-1} x) &= \frac{1}{1+x^2} & \frac{d}{dx}(\cot^{-1} x) &= -\frac{1}{1+x^2}\end{aligned}$$

Integration Formulas:

The following useful integration formulas are obtainable from the above differentiation formulas:

1. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
2. $\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$
3. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

Section 7.8: Indeterminate Forms and L'Hospital's Rule

L'Hospital's Rule:

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval containing the point a (except possibly at a). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \text{ and } \lim_{x \rightarrow a} g(x) = 0$$

or

$$\lim_{x \rightarrow a} f(x) = \pm\infty \text{ and } \lim_{x \rightarrow a} g(x) = \pm\infty$$

That is, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right hand side exists (or is $\pm\infty$).

Indeterminate Products:

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then we call $\lim_{x \rightarrow a} f(x)g(x)$ an *indeterminate form of type $0 \cdot \infty$* .

Convert this form to an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \text{ or } fg = \frac{g}{1/f}$$

Then apply L'Hospital's rule to evaluate the limit.

Indeterminate Differences:

If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$, then $\lim_{x \rightarrow a} [f(x) - g(x)]$ is called an *indeterminate form of type $\infty - \infty$* .

Try to convert $f - g$ to a quotient of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by using a common denominator, rationalization, or factorization. Then apply L'Hospital's rule.

Indeterminate Powers:

We get several indeterminate forms from the limit:

$$\lim_{x \rightarrow a} [f(x)]^{g(x)}$$

The three cases are summarized here:

1. Type 0^0 $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$
2. Type ∞^0 $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = 0$
3. Type 1^∞ $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$

For each case we can rewrite the function using either the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \quad \text{then } \ln y = g(x) \ln f(x)$$

or an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

Here, the product $g(x) \ln f(x)$ is an indeterminate product of type $0 \cdot \infty$ which can be evaluated using the earlier technique.