POLS509 - The Linear Model
January 21, 2003
Limits and Differential Calculus

1 Limits

The idea of limits isn't necessarily an easy one, but its important to understand derivatives and integrals. This is mostly intuitive discussion...

Suppose we're interested in the value of some function $f(X)$ when X is very close to some number a, but not equal to a.

Why would this ever be the case?

- Book Example: Carl Lewis' speed at the 50-meter mark of the race – this is a silly example, since we can just measure it.
- Sometimes we can't ever achieve the value a.
  - Physicists may want to know what happens at zero air pressure – e.g. the reduction of acceleration due to gravity because of air resistance.
    - Can never get there (no perfect vacuum).
    - Measure at successively decreasing levels of air pressure.
    - If the measurement approaches some fixed number as the air pressure drops, we can assume that number is the measurement in a perfect vacuum.
  - Sometimes the function is not defined at the value a.
    - E.g., Interest group utility = quantity of policy, discounted by how much the group agrees with that policy (as determined by the House).
    - E.g. $U_i = \frac{q_i}{\sum(i-H)^2}$.
    - What happens to U as the group and the legislature converge in preferences (i.e., as i gets closer and closer to H)?
    - U is undefined when $(i - H) = 0$, but we can still calculate the limit of the function.

QUESTION: As X gets closer and closer to a (but does not equal a), does $f(X)$ get closer and closer to some number L?

If the answer is yes, then we say "The limit of $f(X)$ as X approaches a is L", or "L is the limit of $f(X)$ at a."

And we write "$\lim_{X \to a} f(X) = L$".

1.1 Calculating Limits

Now this isn't very precise. ("Closer and closer"?) Intuitively, how might we do this?

A: We "close in" on the limit by looking at points (the "neighborhood") around the point we're interested in...

Consider the function $f(X) = \frac{3X-1}{2}$, and suppose we pick $a = 4$. We want to know $\lim_{X\to 4} f(X)$.

- Consider that, if X is close to 4 then $(3X - 1)$ must be close to 11, and so $\frac{3X-1}{2}$ must be close to 5.5...
- Alternately, consider:

$$\begin{array}{ll} f(3.9) = 5.35 & f(4.1) = 5.65 \\ f(3.99) = 5.485 & f(4.01) = 5.515 \\ f(3.999) = 5.4985 & f(4.001) = 5.5015 \\ f(3.9999) = 5.49985 & f(4.0001) = 5.50015 \end{array}$$

etc.

Here, $\lim_{X\to 4} f(X) = 5.5$

Well, we could have just "plugged in" 4 and gotten this, right? Sure, but sometimes we can't...

Ex: find $\lim_{X\to 9} \frac{X-9}{\sqrt{X}-3}$

- We can't substitute 9 into the formula, BUT
- Factoring the numerator yields $\frac{(\sqrt{X}+3)(\sqrt{X}-3)}{\sqrt{X}-3} = \sqrt{X} + 3$.
- Thus, for all X *except* $X = 9$, $f(X) = \sqrt{X} + 3$.
- The closer X is to 9, the closer $f(X)$ is to $\sqrt{9} + 3 = 6$.

This still seems like a lousy way to get limits...

Now reconsider the previous function $f(X) = \frac{3X-1}{2}$. Rewrite:

$$\begin{array}{lll} 5.35 < f(X) < 5.65 & \text{whenever} & 3.9 < X < 4.1 \\ 5.485 < f(X) < 5.515 & \text{whenever} & 3.99 < X < 4.01 \\ 5.4985 < f(X) < 5.5015 & \text{whenever} & 3.999 < X < 4.001 \end{array}$$

etc...

More generally, for all $\delta > 0$ and $\varepsilon > 0$,

$$5.5 - \varepsilon < f(X) < 5.5 + \varepsilon \quad \text{whenever} \quad 4 - \delta < X < 4 + \delta$$

So, we get the first statement if $\delta = 0.1$ and $\varepsilon = 0.15$, etc.

Now subtract 5.5 from the first equality and 4 from the latter, to get:

$$-\varepsilon < f(X) - 5.5 < \varepsilon \quad \text{whenever} \quad -\delta < X - 4 < \delta$$

We can rewrite this in terms of absolute values:

$$|f(X) - 5.5| < \varepsilon \quad \text{whenever} \quad |X - 4| < \delta$$

This reads: "$f(X)$ is within ε units of 5.5 whenever X is within δ units of 4 (but not equal to 4)."

This process is generally reversed to find a limit...

- Start with some small value for ε; this gives us a (small) interval containing $f(X)$.
- Next, we determine if, by forcing X sufficiently close to a, we can get a value of $f(X)$ within that interval.

More formally:

Given any real number $\varepsilon > \mathbf{0}$, , find a real number $\delta > \mathbf{0}$ such that $|\mathbf{f}(\mathbf{X}) - \mathbf{L}| < \varepsilon$ whenever $\mathbf{0} < |\mathbf{X} - \mathbf{a}| < \delta$.

1.2 A Definition

If a function f is defined through an open interval containing a, except possibly a itself, then the limit of $f(X)$ as X approaches a is L if for every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $|f(X) - L| < \varepsilon$ whenever $0 < |X - a| < \delta$.

or...

"$f(X)$ can be made arbitrarily close to L by choosing X sufficiently close to a."

Note that we specify ε first... This is because δs are not unique: once we find a δ which satisfies the criterion, any smaller value of δ will also do so.

1.3 Stuff about limits...

- If a function meets this criterion, then it is said to have a limit at a (the limit **exists**).
- If a limit exists, it is **unique** (don't ask why).

Returning to our example, now to prove that $\lim_{X \to 4} f(X) = 5.5$.

Start with the inequality, "plugging in" the possible values for L:

$$\begin{aligned} &|\tfrac{3X-1}{2} - 5.5| < \varepsilon \\ &|3X - 1 - 11| < 2\varepsilon \\ &|3X - 12| < 2\varepsilon \\ &|3(X-4)| < 2\varepsilon \\ &|X - 4| < \tfrac{2}{3}\varepsilon \end{aligned}$$

So if $\delta = \frac{2}{3}\varepsilon$, then the second inequality is true (for $0 < |X - 4| < \delta$), and thus so is the first.

1.4 One-Sided Limits

Most of the time, the limit of a function doesn't change depending on which "direction" you approach it from.

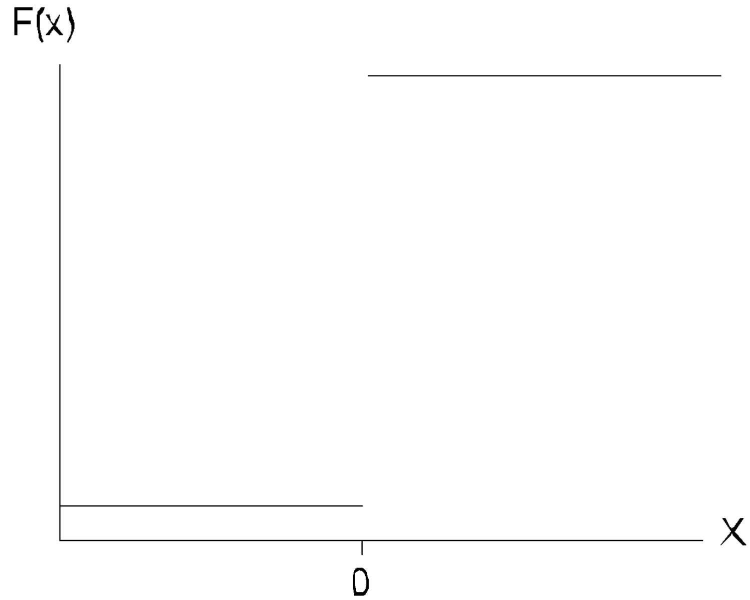
I.e. $\lim_{X \to a} 2X + 1 = 2a + 1$, and this is true irrespective of where you "come at" a from (i.e. from a negative or positive direction).

But this is not always true, though... Take the "Fiesta Bowl Function":

- I bet M dollars on the Fiesta bowl (no point spread or anything like that).
- Define $X = \text{Points}_{Miami} - \text{Points}_{OSU}$.
- Then:

$$\begin{aligned} f(X) &= M \text{ for } X > 0 \\ &= -M \text{ for } X < 0 \\ &= \text{ is undefined for } X = 0. \end{aligned}$$

- That is, its a "step function".



Note: Here, $f(a)$ is undefined at $X = 0$ (but we can still calculate a limit, right?). However...

The limit at $X = 0$ depends on which direction one approaches zero from...

This is written as $\lim_{X \to 0+} f(X)$ and $\lim_{X \to 0-} f(X)$. Here,

- $\lim_{X \to 0+} f(X) = M$
- $\lim_{X \to 0-} f(X) - -M$

In order for a limit at a to exist, $\lim_{X \to a+} f(X)$ and $\lim_{X \to a-} f(X)$ must both exist, and be equal. Thus, for the "Fiesta Bowl Function" here, there is no limit for $f(X)$ at zero.

1.5 Properties of Limits

To motivate this discussion, consider the kinds of functions that we might want to determine a limit for...

1.5.1 Constant functions

- $f(X) = c$: all values of X yield the same value for $f(X)$.
- What is $\lim_{X \to a}$ under this circumstance?

- ∘ First, pick a value for epsilon, giving you the "neighborhood" around $f(X)$.
- ∘ No matter what value you pick for X, $f(X)$ will be in this "neighborhood".
- ∘ Substitute c for L in the definition...
- ∘ Thus, for every $\delta > 0, |f(X) - c| < \varepsilon$ whenever $0 < |X - a| < \delta$.

- **The limit of a constant is the constant**.

Well, that's not very helpful, yet...

1.5.2 More General Rules

If there exist L and M such that $\lim_{X \to a} f(X) = L$ and $\lim_{X \to a} g(X) = M$, then:

$$\begin{aligned} \lim_{X \to a} [f(X) + g(X)] &= \lim_{X \to a} f(X) + \lim_{X \to a} g(X) \\ &= L + M \end{aligned}$$

$$\begin{aligned} \lim_{X \to a} [f(X) \times g(X)] &= \lim_{X \to a} f(X) \times \lim_{X \to a} g(X) \\ &= L \times M \end{aligned}$$

$$\begin{aligned} \lim_{X \to a} \frac{f(X)}{g(X)} &= \frac{\lim_{X \to a} f(X)}{\lim_{X \to a} g(X)} \\ &= \frac{L}{M} \end{aligned}$$

That is,

- *The limit of a sum is the sum of the limits.*
- *The limit of a product is the product of the limits.*
- *The limit of a quotient is the quotient of the limits.*

By extension, if k is some real constant, then:

$$\begin{aligned} \lim_{X \to a} [k \times f(X)] &= \lim_{X \to a} k \times \lim_{X \to a} f(X) \\ &= k \times \lim_{X \to a} f(X) \\ &= k \times L \end{aligned}$$

That is, we can "pull through" a constant term. Additionally, the special case of $k = -1$ lets us also say that *the limit of a difference is the difference of the limits.*

We can also use the multiplication rule to talk about *exponents*:

$$\begin{aligned}
\lim_{X \to a} [f(X)^n] &= \lim_{X \to a} [f(X) \times f(X) \times ...] \\
&= \lim_{X \to a} f(X) \times \lim_{X \to a} f(X) \times ... \\
&= [\lim_{X \to a} f(X)]^n \\
&= L^n
\end{aligned}$$

Moreover, for the case where $f(X) = X$, note that:

$$\begin{aligned}
\lim_{X \to a} [f(X)^n] &= [\lim_{X \to a} X]^n \\
&= a^n
\end{aligned}$$

These various things let us prove some very useful theorems. For example, we can show that:

If $a \in \mathbb{R}$ and $f(X)$ is a polynomial function, then $\lim_{X \to a} f(X) = f(a)$.

- Essentially, this means that the limit of a polynomial function can be found via substitution of a for X.
- Proof:
  - Begin with $f(X) = b_n X^n + b_{n-1} X^{n-1} + ... + b_1 X + b_0$.
  - Then:

$$\begin{aligned}
\lim_{X \to a} f(X) &= \lim_{X \to a} (b_n X^n) + \lim_{X \to a} (b_{n-1} X^{n-1}) + ... + \lim_{X \to a} (b_1 X^1) + \lim_{X \to a} (b_0 X^0) \\
&= b_n \lim_{X \to a} (X^n) + b_{n-1} \lim_{X \to a} (X^{n-1}) + ... + b_1 \lim_{X \to a} (X) + b_0 \lim_{X \to a} (1) \\
&= b_n a^n + b_{n-1} a^{n-1} + ... + b_1 a + b_0 \\
&= f(a)
\end{aligned}$$

The same is true for rational functions...

- If $f(X)$ is any rational function, then $\lim_{X \to a} f(X) = f(a)$.
- This is true because of the division rule.

1.6 Limits at Infinity and Infinite Limits

1.6.1 Limits at Infinity

We sometimes want to know what happens to a function as X increases or decreases without limit.

- Consider $f(X) = 2 + \frac{1}{X}$...
- What happens as x increases without limit?
- A: $f(X) \to 2$.
- Here, we write $\lim_{X\to\infty} f(X) = 2$

Generally speaking:

- If a function is defined on an open interval (c, ∞), then $\lim_{X\to\infty} f(X) = L$ implies that for every $\varepsilon > 0$ there is a corresponding number N such that $|f(X) - L| < \varepsilon$ whenever $X > N$.
- Put more simply, we can make the function $f(X)$ arbitrarily close to L by increasing the value of X without bound.
- The same is true on the negative side: If a function has a limit as $X \to \infty$, then we can make $f(X)$ arbitrarily close to that limit by decreasing the value of X without bound.

1.6.2 Infinite Limits

In addition to being interested in X going to infinity, we also encounter functions which themselves go to infinity for certain values...

Reconsider the interest group utility function from earlier:

$$U_i = \frac{q_i}{\sum_{i=1}^{N}(i - H)^2}$$

Assess $\lim_{i\to H} U_i$.

- The denominator goes to zero, and so
- $U_i \to \infty$.
- We thus would say that $\lim_{i\to H} U_i = \infty$.
- More generally, a function has a (positive) infinite limit at a if $f(X)$ can be made arbitrarily large by setting X successively closer to a.

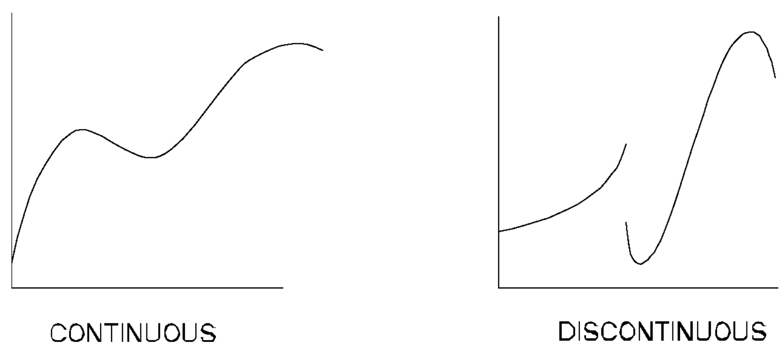
- ∘ If $\lim_{i \to H} f(X) = \pm\infty$, then the line $X = a$ is called a *vertical asymptote* of $f(X)$.
- ∘ We also say that $f(X)$ has an *infinite discontinuity* at a.

What's a discontinuity?...

2 Continuity and Continuous Functions

If $\lim_{X \to a} f(X) = f(a) \forall a$, then $f(X)$ is said to be *continuous*.

- Intuitively, it has no "gaps" or "holes" in the function.
- E.g. a "step function" (like that for Bowl outcomes) is *not* continuous.



Technically, for a function to be continuous at a point a, it must meet three conditions:

1. $f(a)$ is defined (i.e. $f(X)$ is defined at a).
2. $\lim_{X \to a} f(X)$ exists.
3. $\lim_{X \to a} f(X) = f(a)$.

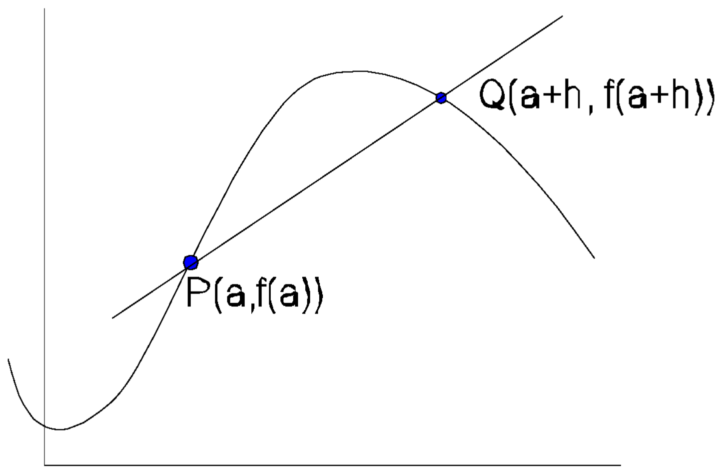
If these conditions are met for all a in the function's domain, then the function is continuous. Note that a function can also be continuous over a specifically defined interval...

The idea of continuity brings us to...

3 Differential Calculus

The *derivative* of a function.

Consider a function $f(X)$ in $\mathbb{R}$.



- We can represent a point a on the graph of the function by the pair $(a, f(a))$ (call this point P).
- We can write another point Q as $(a+h, f(a+h))$.
- We can write the slope of the line through P and Q as:

$$\begin{aligned} m &= \frac{Y_2 - Y_1}{X_2 - X_1} \\ &= \frac{[f(a+h) - f(a)]}{((X+h) - X)} \\ &= \frac{[f(a+h) - f(a)]}{h} \end{aligned}$$

Now what happens as $h \to 0$?

A: The line becomes the tangent to $f(X)$ at a.

So one way to define the slope of a line tangent to the function $f(X)$ at a is:

$$m(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

...provided that this limit exists.

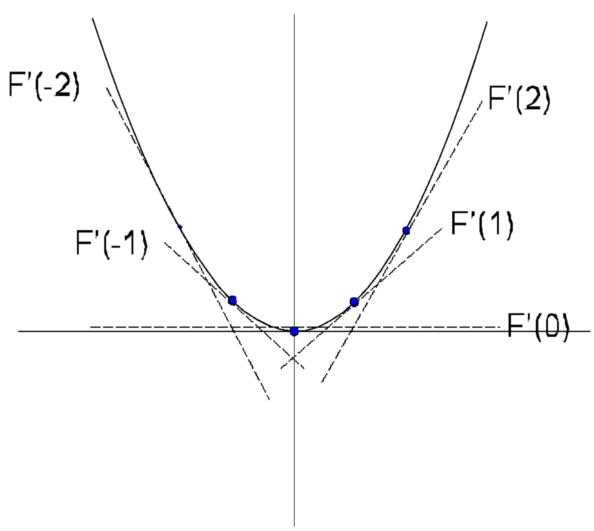
Intuitively, this is what a derivative is: **the slope of a line tangent to a function.**

As a simple illustrative example, consider the function $f(X) = X^2$.

We can calculate a few values of this:

X	$f(X)$	$\frac{\partial}{\partial X} f(X)$
-2	4	-4
-1	1	-2
0	0	0
1	1	2
2	4	4

And we can graph them as well:



So we can think of a derivative as the function mapping values of X to the slope of the line tangent of the function at X.

3.1 A Bit More Math

In general,

$$\frac{\partial}{\partial X} f(X) = \lim_{h \to 0} \frac{f(X+h) - f(X)}{h}$$

Various notations for this include:

- $\frac{\partial}{\partial X} f(X)$,
- $\frac{df}{dX}$,
- $f'(X)$, etc.

I'll generally use the first of these...

We say "The derivative of $f(X)$ with respect to X" (or the "first derivative of $f(X)$").

If this limit exists, we say "f is differentiable".

Note that this also implies that $f(X)$ is continuous (but not all continuous functions are differentiable...).

Another way of thinking about the derivative is the *instantaneous rate of change* in a function; i.e., the extent to which a function is changing at precisely that point...

Let's do an example:

If $f(X) = 3X^2 - 5X + 4$, find $\frac{\partial}{\partial X} f(X)$...

$$\begin{aligned}
\frac{\partial}{\partial X} f(X) &= \lim_{h \to 0} \frac{f(X+h) - f(X)}{h} \\
&= \lim_{h \to 0} \frac{[3(X+h)^2 - 5(X+h) + 4] - (3X^2 - 5X + 4)}{h} \\
&= \lim_{h \to 0} \frac{(3X^2 + 6Xh + 3h^2 - 5X - 5h + 4) - (3X^2 - 5X + 4)}{h} \\
&= \lim_{h \to 0} \frac{6Xh + 3h^2 - 5h}{h} \\
&= \lim_{h \to 0} 6X + 3h - 5 \\
&= 6X - 5
\end{aligned}$$

This seems like a pain.

There must be an easier way...

3.2 Rules for finding derivatives in one variable...

First, some basic results...

- If $f(X) = c$, then $\frac{\partial}{\partial X} f(X) = 0$.
  - The derivative of a constant function is zero.
  - Intuitively: Draw a line "tangent" to a constant function. What is its slope? (A: Zero)
  - Or, if the derivative is the instantaneous rate of change, then for a constant, there is none.
- If $f(X) = mX + b$, then $\frac{\partial}{\partial X} f(X) = m$.
  - The derivative of a linear function is its slope.
  - Again, consider a "tangent" line to the function... (Its slope is same as that of the line).
  - What is the instantaneous rate of change for a linear function? (A: its slope).

Now, a few more general things:

- $\frac{\partial}{\partial X} c \times f(X) = c \times \frac{\partial}{\partial X} f(X)$.
  - The derivative of a constant times a function is equal to the constant times the derivative of the function.
  - That is, you can "pull a constant through" differentiation.
- if $f(X) = X^n$, then $\frac{\partial}{\partial X} f(X) = nX^{n-1}$.
  - The derivative of the function X^n is n times X^{n-1}.
  - This is known as the **power rule**.
  - Relatedly: if $f(X) = X^{-n}$, then $\frac{\partial}{\partial X} f(X) = -nX^{-n-1}$ (That is, its also true for negative (integer) exponents).
  - This means that, in general for polynomials, the first derivative of a polynomial of order k will be of order $k - 1$.
- $\frac{\partial}{\partial X} [f(X) + g(X)] = \frac{\partial}{\partial X} f(X) + \frac{\partial}{\partial X} g(X)$.
  - The derivative of the sum of two functions is equal to the sum of their derivatives.
  - That one's easy enough to understand...
- $\frac{\partial}{\partial X} [f(X) \times g(X)] = \left[\frac{\partial}{\partial X} f(X) \times g(X)\right] \times \left[\frac{\partial}{\partial X} g(X) \times f(X)\right]$.

- The derivative of the product of two functions is equal to the product of the derivative of the first times the second, plus the product of the derivative of the second times the first.
- This is the **Product Rule** for differentiation.

- $$\frac{\partial}{\partial X}\frac{f(X)}{g(X)} = \frac{\left[\frac{\partial}{\partial X}f(X) \times g(X)\right] - \left[\frac{\partial}{\partial X}g(X) \times f(X)\right]}{g(X)^2}.$$
  - The derivative of the quotient of two functions is equal to the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all divided by the denominator squared.
  - Never mind why (consult a calculus book, if you really, really need to know...).

- $\frac{\partial}{\partial X}f(g(X)) = \frac{\partial}{\partial X}g(X) \times \frac{\partial}{\partial X}f(g(X))$
  - The derivative of the composition of two functions is equal to the derivative of the second function times the derivative of the first function evaluated at the second function.
  - This is the Chain Rule for differentiation.
  - It is also very important, and useful.